مزاوجة طريقة التنظيم مع طريقة الحساب المباشر في حل معادلة فردهولم التكاملية في بعدين

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الملخص :

في هذه الورقة، سوف نستخدم مزاوجة بين طريقتي التنظيم والحساب المباشر لحل معادلة (فردهولم) التكاملية الخطية من النوع الأول وفي بعدين عن طريق تحويلها إلى النوع الثاني أولا : باستخدام طريقة التنظيم ، بعد ذلك نستخدم طريقة الحساب المباشر لحل المعادلة الناتجة ؛ كما نقدم بعض الأمثلة لتوضيح وإثبات صحة هذه الطريقة.

A coupling Method of Regularization and Direct Computation Method for solving Two-dimensional Fredholm Integral Equations

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Abstract:

In this paper, we will use the combination of Regularization method and Direct computation method, or shortly, Regularization-Direct method for solve two dimensional linear Fredholm integral equations of first kind, by converting the first kind of equation to the second kind by applying the regularization method. Then the Direct compotation method is applying to getting the resulting second kind of equation to obtain a solution. A few examples are provided to prove the validity and applicability of this approach.

Keywords: linear integral equations; Fredholm integral equations; Regularization method**,** direct compotation method**;** two- dimensional integral equations.

1. Introduction:

In recent years, several methods have been used to approximate the solution of one and two- dimensional linear integral equations of the first kind by researchers in mathematics, physics and engineering. An algorithm to handle the linear Fredholm integral equations of the first kind was proposed by Wazwaz [4]. Whilst Bazrafshan et al. [12] used homotopy analysis method to solve two-dimensional integral equations. Ahmed Altürk [1] have been used the regularization-homotopy method for solving two dimensional Fredholm integral equation of the first kind. In this paper, we consider Linear Fredholm integral equations of first kind in two dimensional of the form

$$\int_a^b \int_c^d K(x,t,y,z)\, u(y,z)\, dydz = f(x,t), \qquad x \in [a,b], t \in [c,d] \qquad (1)$$

Where $f(x,t)$, and the kernel $K(x, t, y, z)$ are continuous and given functions. Research on the two-dimensional case has been getting more attention recently [2, 3], [10],[12, 13], [16, 17], and [19]

In this work, we will apply the regularization method [1] that is considered reliable especially in solving first kind integral equations. The method transforms a first kind equation to a second kind equation. By converting the first kind to a second kind, then we can apply the existing techniques of the second kind to the transformed equation. The direct computation method will be combined with the regularization method to handle Fredholm integral equations of the first kind.

2. The Regularization Method:

The regularization method was established independently by Tikhonov [7, 8], Phillips [11]. This method consists of transforming the first kind integral equations to the second equations. The details for one dimensional case can be found in [5, 7, 8], and [11]. We instead focus on the two-dimensional case. The regularization method for the two-dimensional Fredholm integral equations of the first kind was introduced in [13]. We now briefly explain the method. Like in one dimensional case, the regularization method transforms the two-dimensional Fredholm integral equation of the first kind (1) to a two-dimensional linear Fredholm integral equation of the second kind as

$$\mu u_\mu(x,t) = f(x,t) - \int_a^b \int_c^d K(x,t,y,z) u_\mu(x,t)\, dy dz \qquad (2)$$

Where μ is a small positive parameter. Notice that one could express equations (2) as

$$u_\mu(x,t) = \frac{1}{\mu} f(x,t) - \frac{1}{\mu} \int_a^b \int_c^d K(x,t,y,z) u_\mu(y,z)\, dy dz \qquad (3)$$

It was shown in [9] that the solution of equation (3) as $\mu \to 0$ approaches $u(x, t)$ which is the solution of equation (1). In other words,

$$u(x,t) = \lim_{\mu \to 0} u_\mu(x,t)$$

3. Existeness and uniqueness solution:

In this section we state some results about existence and uniqueness solution from the operator theory [15, 18]. Let

$$A: \mathrm{C}([a,b] \times [c,d]) \to \mathrm{C}([a,b] \times [c,d])$$

And the integral operator

$$Au(x,t) = \int_c^d \int_a^b K(x,t,y,z)u(y,z)dydz \qquad x \in [a,b]\,, t \in [c,d] \qquad (4)$$

Theorem1. Let $k: \mathrm{C}([a,b] \times [c,d] \times [a,b] \times [c,d]) \to \mathbb{R}$ be continuous, then the operator A defined by (4) is bounded with the norm:

$$||A||_\infty = \max_{x \in [a,b], t \in [c,d]} \int_c^d \int_a^b |k(x,t,y,z)|dydz$$

Proof. See [15].

Theorem2.

For any $\mu > 0, f \in C([a,b] \times [c,d])$ and bounded linear operator A, the equation (3) is solvable and has a unique solution.

Proof. See [9].

We know that the two-dimensional Fredholm integral equations of the first kind are ill-posed problems. The solution for an ill-posed problem may not exist, and if it exists it may be non-unique [13]. We will apply the regularization method to convert the first kind Fredholm integral equation to the second kind integral equation. Then, the resulting second kind integral equation will be solved by the direct compotation technique that will be presented in the next section.

4. The Direct compotation method:

The direct computation method will be applied to solve the Fredholm integral equations

$$u(x,t) = f(x,t) + \int_c^d \int_a^b k(x,t,y,z)u(y,z)dydz \qquad (5)$$

. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$K(x,t,y,z) = \sum_{i=1}^{n} g_i(x,t)h_i(y,z) \qquad (6)$$

To apply the direct computation method to (5), substituting from (6) to (5) implies

$$u(x,t) = f(x,t) + \sum_{i=1}^{n} g_i(x,t) \int_c^d \int_a^b h_i(y,z)u(y,z)dydz$$

Each integral at the right side depends only on the variables y and z, so by setting

$$\alpha_i = \int_c^d \int_a^b h_i(y,z)u(y,z)dydz, \quad i = 1,2,\dots,n, \qquad (7)$$

We obtain

$$u(x,t) = f(x,t) + \sum_{i=1}^{n} \alpha_i g_i(x,t) \qquad (8)$$

Substituting (8) into (7) yields

$$\alpha_i = \int_c^d \int_a^b h_i(y,z)\, f(y,z)dydz \; + $$

$$+\sum_{j=1}^{n} \int_c^d \int_a^b \left(\alpha_j h_i(y,z) g_j(y,z)\right) dydz\,, \quad i = 1,2,\dots,n, \qquad (9)$$

Which is a system of n algebraic equations with unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$, so by the System (9) and using the obtained numerical values of $\alpha_1, \alpha_2, \dots, \alpha_n$ into (8), the solution $u(x,t)$ of the Fredholm integral equation (5) is obtained.

5. Regularization - Direct Computation Method:

We investigate a two-dimensional linear Fredholm integral equation of the second kind:

$$f(x,t) = \int_c^d \int_a^b k(x,t,y,z)u(y,z)dydz \qquad (10)$$

We give an algorithm about how to apply the method.

We recall from Section 2 that the regularization method transform Equation (10) to the following equation:

$$u_\mu(x,t) = \frac{1}{\mu} f(x,t) - \frac{1}{\mu} \int_c^d \int_a^b k(x,t,y,z) u_\mu(y,z) dydz \qquad (11)$$

Since the aim of this article is to extend the direct computation-regularization method introduced in [1], we construct the direct computation as follows:

$$u_\mu(x,t) = \frac{1}{\mu} f(x,t) + \frac{1}{\mu} \sum_{i=1}^{n} g_i(x,t) \int_c^d \int_a^b h_i(y,z) u_\mu(y,z) dydz$$

$$u_\mu(t,x) = \frac{1}{\mu} f(x,t) + \frac{1}{\mu} \sum_{i=1}^{n} \alpha_i g_i(x,t) \qquad (12)$$

$$\alpha_i = \int_c^d \int_a^b h_i(y,z) u_\mu(y,z) dydz \quad , i = 1,2, \dots . n \qquad (13)$$

6. Illustrative Examples

We note that we assume the kernel k(x, t, y, z) is separable, i.e. $k(x,t,y,z) = g(x,t)h(y,z)$. We also require the function $f(x,t)$ involve components matched by $g(x,t)$. This is a necessary condition for a solution to exist (10).

Example 1:

Consider the following linear Fredholm integral equation of the first kind:

$$(x+t)=\int_0^1\int_0^1 (x+t)y\,u(y,z)dydz \tag{14}$$

The regularization method transforms Equation (23) to

$$u_\mu(x,t)=\frac{1}{\mu}(x+t)-\frac{1}{\mu}\int_0^1\int_0^1 (x+t)y\,u_\mu(y,z)dydz \tag{15}$$

Now, we solve (15) by the direct computation method. To do this, equation (15) can be written as

$$u_\mu(x,t)=\frac{1}{\mu}(x+t)-\frac{1}{\mu}(x+t)\alpha \tag{16}$$

Where α is constant given by

$$\alpha=\int_0^1\int_0^1 yu_\mu(x,t)dydz \tag{17}$$

Substituting (16) into (17) implies

$$\alpha=\frac{7}{7+12\mu} \tag{18}$$

So, when $\mu\to 0$ we obtain

$$u(x,t)=\frac{12}{7}(x+t)$$

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