On The 2D Euler-Stratified System in The $LBMO$ Space

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Abstract:

In this paper, we will prove the unique global solution for the Euler-stratified system with bounded vorticity in the $LBMO$ spaces.

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1. Introduction.

Euler stratified (also nameed, Euler Boussinesq) equations for the incompressible fluid flows in $\mathbb{R}^2$ is of the form

$$(1.1)\qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = \theta e_2, \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, \\ div\ v = 0, \quad (v,\theta)|_{t=0} = (v_0,\theta_0), \end{cases}$$

where $v = (v_1, v_2), v_j = v_j(x,t),\ j = 1,2,\ (x,t) \in \mathbb{R}^2 \times [0,\infty[$ is the velocity field, $p = p(x,t)$ is the scalar pressure, the function $\boldsymbol{\theta}$ is the scalar temperature and $e_2 = (0,1)$. Note that the system (1.1) coïncides with the classical incompressible Euler equations when the initial temperature $\boldsymbol{\theta}_0$ is identically constant. Recall that Euler system is given by,

$$(1.2)\qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = 0, \\ div\ v = 0, \\ v|_{t=0} = v_0 \end{cases}$$

The question of local well-posedness of (1.2) with smooth data was resolved by many authors in different spaces, see for example [6,7]. In this context, the vorticity $\omega = curl\ v$ play a fundamental role. In fact, the well-known BKM criterion [4] ensures that the development of finite time singularities for these solutions is related by the blow-up of the L^∞ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of the two-dimensional Euler solutions with smooth initial data, since the vorticity satisfies the equation.

$$\partial_t \omega + v.\nabla\omega = 0,$$

and then all the L^p norms are conserved. The global well posedness result for the system (1.2), was proved in different spaces and we focus on the paper of Bernicot and keraani in [1], where the authors proved a similar result in the $LBMO$ spaces.

We turn now to the system (1.1) and note that the global well-posedness result for this system was solved by many authors and in a different functional spaces, see for instance [2,5].

In this paper, we extend a global well-posedness result for the system (1.1) with bounded vorticity in the $LBMO$ spaces and we will use the same approach of [1]. Our main result is the following.

Theorem 1.1 Assume $p \in]1,2[$. Let v_0 be a divergence-free vector field of vorticity $\omega_0 \in L^p \cap LBMO$ and let $\theta_0 \in L^2$ a real-valued function. Then there exists a unique global weak solution (v, θ) of the system (1.1). Moreover, there exists a constant C_0 depending only on the $L^p \cap LBMO$ norm of ω_0 and the L^2 norm of θ_0 such that

$$(1.3) \dots\dots\dots. \|\omega(t)\|_{L^p \cap LBMO} \leq C_0 e^{C_0 t}.$$

The vorticity for the system (1.1) satisfies the equation

$$\begin{cases} \partial_t \omega + v.\nabla\omega = \partial_1 \theta, \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0 \end{cases}$$

Taking the L^2 scalar product, we get successively

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \int_0^t \|\partial_1 \theta(\tau)\|_{L^2} d\tau,$$

and

$$\|\theta(t)\|_{L^2} + \|\nabla\theta(t)\|_{L^2} = \|\theta_0\|_{L^2}.$$

We use to prove (1.3)a logarithmic estimate in the space $L^p \cap LBMO$ see Theorem 2 in [1], that we recall in section 3 (Theorem 3.2 in this paper).

The paper is organized as follows. In section 2, we recall some functional spaces. Section 3 is devoted to recall some properties of the $LBMO$ spaces and in section 4, we prove our Theorem 1.1.

2. Functional spaces

In this preliminary section, we are going to recall some functional spaces.

Definition 2.1 We define the space LL by the space of the set of bounded vector fields v such that

$$\|v\|_{LL} := sup_{x\neq y}\frac{|v(x)-v(y)|}{|x-y|(1+|\ln(|x-y|)|)} < \infty.$$

Definition 2.2 For every homeomorphism ψ, we set

$$\|\psi\|_{*} := sup_{x\neq y}\Phi(|\psi(x)-\psi(y)|,|x-y|)$$

where Φ is defined on $]0,\infty[\times]0,\infty[$ by

$$\Phi(\mathrm{r},\mathrm{s}) = \begin{cases} \max\left\{\frac{1+|\ln s|}{1+|\ln r|},\frac{1+|\ln r|}{1+|\ln s|}\right\}, & if\ (1-s)(1-r)\geq 0 \\ (1+|\ln s|)(1+|\ln r|)\,, & if\ \ (1-s)(1-r)\leq 0. \end{cases}$$

Since Φ is symmetric, then $\|\Phi\|_{*} = \|\psi^{-1}\|_{*} \geq 1$. It is clear also that every homeomorphism Φ satisfying

$$\frac{1}{C}|x-y|^{\alpha} \leq |\psi(x)-\psi(y)| \leq C|x-y|^{\beta},$$

for some $\alpha,\beta,C > 0$ has its $\|\psi\|_{*}$ finite.

The following proposition was proved in [1].

Proposition 2.3 Let v be a smooth divergence-free vector fields and ψ its flow;

$$\begin{cases} \partial_t\psi(t,x) = v(t,\psi(t,x)) \\ \psi(0,x) = x. \end{cases}$$

Then for every $t \geq 0$ we have

$$\|\psi(t,.)\|_* \leq \exp\left(\int_0^t \|v(\tau)\|_{LL} d\tau\right).$$

3. The $LBMO$ space

In this section, we will state some properties of the $LBMO$ spaces. We first give the following which its proof can be found in [1].

Proposition 3.1 The following properties hold true :

(1) The space $LBMO$ is a Banach space included in BMO and strictly containing $L^\infty(\mathbb{R}^2)$.

(2)For every $g \in C_0^\infty(\mathbb{R}^2)$ and $f \in LBMO$, we have

$$\|g * f\|_{LBMO} \leq \|g\|_{L^1} \|f\|_{LBMO}.$$

The following theorem is the main ingredient for proving Theorem 1.1, see [1] for a proof.

Theorem 3.2 There exists a universal constant $C > 0$ such that,

$$\|f \circ \psi\|_{LBMO \cap L^p} \leq C \ln(1 + \|\psi\|_*) \|f\|_{LBMO \cap L^p}$$

for any Lebesgue measure preserving homeomorphism ψ.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 can be divisid in three steps. In the first step, we will prove some a priori estimates which the main ingredient for the proof of our main result. The second and the third steps deal with the existence and the uniqueness for the solutions. We only prove the first step and the second and the third steps are standard, see [1], [3], [6], [8], [9] and [10].

4.1 A priori estimates. We will prove the following proposition.

Proposition 4.1

Let (v, θ) be a smooth solution of the system (1.1) with vorticity ω. Then there exists a constant C_0 depending only on the norm $L^p \cap LBMO$ of ω_0 and the L^2 norm of θ_0 such that

$$\|v(t)\|_{LL} + \|\omega(t)\|_{L^p \cap LBMO} \leq C_0 e^{C_0\, t}.$$

Proof. We have $\omega(t,x) = \omega_0(\psi_t^{-1}(x))$, where ψ_t is the flow associated to the velocity v. Since v is smooth then $\psi_t^{\pm 1}$ is Lipschitzian for every $t \geq 0$. This implies in particular that

$\|\psi_t^{\pm 1}\|_*$ is finite for every $t \geq 0$. By applying Theorem 3.2 and Proposition 2.3 yield together

$$\begin{aligned}\|\omega(t)\|_{LBMO} &\leq C\|\omega_0\|_{LBMO \cap L^p} \ln(1 + \|\psi_t^{-1}\|_*) \\ &\leq C\|\omega_0\|_{LBMO \cap L^p} \ln(1 + \exp(\int_0^t \|v(\tau)\|_{LL} d\tau)) \\ &\leq C_0\left(1 + \int_0^t \|v(\tau)\|_{LL} d\tau\right).\end{aligned}$$

On the other hand, we have from [3], that

$$\|v(t)\|_{LL} \leq \|\omega(t)\|_{L^2} + \|\omega(t)\|_{B^0_{\infty,\infty}} \quad \ldots\ldots\ldots\ldots (4.1)$$

To estimate $\|\omega(t)\|_{L^2}$, we use first the equation of ω :

$$\partial_t \omega + v.\nabla\omega = \partial_1\theta, \qquad \omega(t=0) = \omega_0$$

Then we have

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\nabla\theta\|_{L^1_t L^2} \quad \ldots\ldots (4.2)$$

what remains then is to estimate $\|\nabla\theta\|_{L^1_t L^2}$. For this purpose, we take the scalar product of the second equation of (1.1) with θ in L^2space. Then the incompressibility condition $div\, v = 0$ leads to the following energy estimate :

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2+2\|\nabla\theta\|_{L^2}^2=0.$$

Then

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2+4\|\nabla\theta(t)\|_{L^2}^2=0.$$

Integrating in time this last differential equation, we get

$$\|\theta(t)\|_{L^2}^2+4\|\nabla\theta\|_{L_t^1L^2}^2=\|\theta_0\|_{L^2}^2.$$

Therefore,

$$\|\nabla\theta\|_{L_t^1L^2}^2\lesssim\|\theta_0\|_{L^2}^2.$$

This implies that

$$\|\nabla\theta\|_{L_t^1L^2}\lesssim\|\theta_0\|_{L^2}$$

This gives in (4.2) that

$$\|\omega(t)\|_{L^2}\lesssim\|\omega_0\|_{L^2}+\|\theta_0\|_{L^2}.$$

Plugging this last estimate in (4.1) and using the embedding $BMO\hookrightarrow B_{\infty,\infty}^0$, we obtain

$$\begin{aligned}\|v(t)\|_{LL}&\leq C(\|\omega_0\|_{L^2}+\|\theta_0\|_{L^2}+\|\omega(t)\|_{BMO}\\&\leq C_0\left(1+\int_0^t\|v(\tau)\|_{LL}\right)d\tau.\end{aligned}$$

Gronwall's lemma gives that

$$\|v(t)\|_{LL}\leq C_0e^{C_0t}.$$

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