Viscous Fluid Flow Around an Elliptic Cylinder Using Finite Element Method

Zinab M. Maatoug & Bashir W. Sharif
Dept. of Mathematics, Faculty of Education, Tripoli University

Abstract:

In this paper we implement the Finite Element Method (FEM) to find the possible solution of Laplace/Poisson equation in two dimension viscous flow. We will follow almost the same steps that have been taken by Dambaru D. Bhatta's paper "Fluid Flow around an Elliptic Cylinder using Finite Element Method" which published in journal of mathematical science and mathematics education [JMS & ME]. The fluid under the investigation is incompressible in term of velocity potential and stream function. Since Laplace/Poisson's equations are involved, then we used Dirichlet and Neumann boundary condition as a boundary value problem (BVP). The triangular element is used to obtain the FEM solution. Finally, due to lack of math lab or other mathematical

soft-wears problems solver to solve our equations, we predicted the nearest solution to our problem.

Key words *: viscous flow, FEM, incompressibility.*

Mathematical Formulation

We start with two dimensional boundary value problem given by well known Laplace's and Poisson's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad and \quad \nabla^2 u + f = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f = 0 \qquad 1$$

Respectively. These equations have many use in applied mathematics, and engineering ([1],…,[8]).

Fluid Flow equations

We know that the mass of conservation principle is :

[Rate of mass accumulation within control volume (CV)]=[Rate of mass flow into CV]-[Rate of mass flow out of CV].

For two-dimensional CV of dimensions Δx and Δy as shown in figure (1) we have

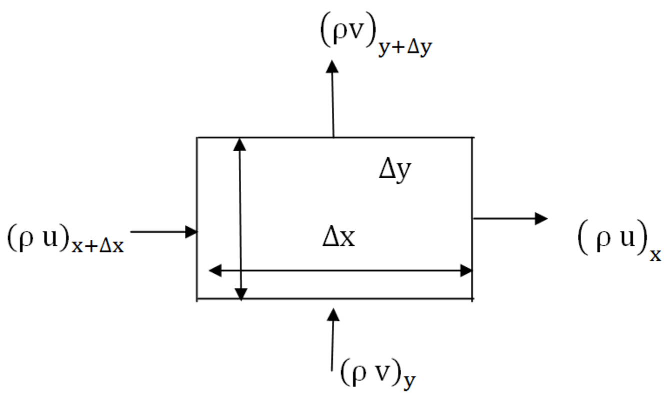

Figure 1 Two-dimensional CV

Mass accumulation rate= $\frac{\partial(\rho\Delta x\Delta y)}{\partial t}$

Mass inflow= $(\rho\ u)_x\Delta y + (\rho\ v)_y\Delta x$

Mass outflow= $(\rho\ u)_{x+\Delta x}\Delta y + (\rho\ v)_{y+\Delta y}\Delta x$

The mass of conservation equation thus gives

$$\frac{\partial(\rho\Delta x\Delta y)}{\partial t} = (\rho u)_x\Delta y + (\rho\ v)_y\Delta x - (\rho u)_{x+\Delta x}\Delta y - (\rho\ v)_{y+\Delta y}\Delta x \qquad 2$$

Dividing by $\Delta x\ \Delta y$ and rearrangement leads to

$$\frac{\partial\rho}{\partial t} = \frac{(\rho u)_x - (\rho\ u)_{x+\Delta x}}{\Delta x} + \frac{(\rho v)_y - (\rho\ v)_{y+\Delta y}}{\Delta y} \qquad 3$$

In the limit as $\Delta x, \Delta y \to 0$, the CV becomes infinitesimally small, and using Taylor series expansions we have

$$(\rho\ u)_{x+\Delta x} \to (\rho\ u)_x + \Delta x\frac{\partial(\rho\ u)}{\partial x}$$

$$(\rho\ v)_{y+\Delta y} \to (\rho\ v)_y + \Delta y\frac{\partial(\rho\ v)}{\partial y}$$

Substituting these results into equation (3) gives

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho\ u)}{\partial x} - \frac{\partial(\rho\ v)}{\partial y} \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \frac{\partial(\rho\ u)}{\partial x} + \frac{\partial(\rho\ v)}{\partial y} = 0$$

Thus the equation of continuity is given by

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0 \qquad 4$$

Here we have $\boldsymbol{u} = (u, v)$ represents the velocity vector of flow filed and ρ is the density of the fluid. If the fluid is incompressible, then ρ is constant, and the equation of continuity becomes $\nabla\cdot\boldsymbol{u} = 0$.

Furthermore, if the motion is irrotational, we have $\nabla\times\boldsymbol{u} = 0$

It is well known in this case there exists a scalar function φ called velocity potential function, such that $\boldsymbol{u} = \nabla\cdot\varphi$.

Now the velocity components (u, v) of $\boldsymbol{u}$ are given by

$$u = \frac{\partial \varphi}{\partial x} \quad and \ v = \frac{\partial \varphi}{\partial y} \ .$$

We also know that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \varphi) = \boldsymbol{\nabla}^2 \varphi$$

and thus φ satisfies the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \ . \qquad 5$$

If we introduce a stream function by ψ, then the relation between velocity potential φ and ψ is given by

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial x} \quad and \ \ v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \ ,$$

Clearly, we see that a stream function is also satisfied by Laplace equation

$$\boldsymbol{\nabla}^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad 6$$

Problem Formulation

Our problem consists of two dimensional boundary value problem (BVP) given by Laplace equation and Poisson's equations.

$$\boldsymbol{\nabla}^2 \cdot u(x, y) + f(x, y) = 0 \qquad in \quad D \qquad 7$$

$$u = g(s) \qquad on \quad \Gamma_1 \qquad 8$$

$$\frac{\partial u}{\partial n} + \alpha(s) u = h(s) \qquad on \quad \Gamma_2 \qquad 9$$

Here we follow the same steps taken by Dambaru D. Bhatta's for the domain and boundary condition in his paper. Therefore, D is the interior of the domain, and Γ_1 and Γ_2 are the boundary sets for the domain D such that $D = \Gamma_1 \cup \Gamma_2$. The boundary condition on Γ_1 means the Dirichlet boundary condition and Γ_2 is referenced to Robins boundary condition. If $\alpha = 0$, equation 9 becomes Neumann boundary condition as shown in figure 2.

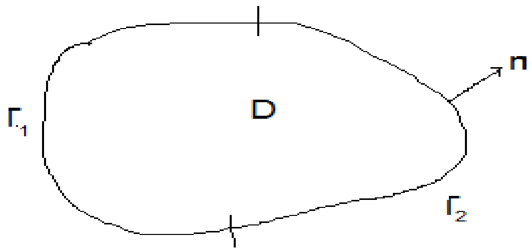

Figure 2: The domain D of the Laplace-Poisson equation with Dirichlet boundary Γ_1 and Neumann boundary Γ_2

Using Finite Element Method to Obtain the Solutions

The following steps indicate the formulation of the problem and to set boundary value problem.

1- Disecting the Domain

The first step is to divide the whole domain into finite parts or elements, not necessarily equal to each other. Some authers call this process a Finite Element Method (FEM). The parts are constructed by joining the nodes with each other by straight line to form triangulars. Here, we used linear triangular to minimizing the whole domain. Then, we numbered these parts and the nodes that join the triangular for our calculations. So there will be area called domains D^e that contains the node and bounded by Γ^e.

2- Weak formulation

The weak formulation of a differential equation is a weighted integral- statement that is equivalent to both the governing differential equation as well as certain types of boundary conditions. We shall develop the weak form of equation (7) over the typical element D^e. First, we take all nonzero expressions in equation (7) to one side of the equality, multiply the resulting equation by a test function $w(x,y)$, then integrate the resulting equation over the element D^e as follows:

$$\iint_{D^e} w(x,y)[-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)-f(x,y)]dA=0 \qquad 10$$

Then we distribute the test function $w(x,y)$ so that both u and w are required to be differential only once with respect to x and y. To obtain the weak formulation, we use the following theorem (Green-Gauss)

$$\iint_{D^e} \nabla\cdot\boldsymbol{u}\ dA=\int_{\Gamma^e}\boldsymbol{n}\cdot\boldsymbol{u}\,ds \qquad 11$$

Where $\boldsymbol{u}$ is a vector field and the line integral is evaluated in a counter-clockwise sense around the boundary curve Γ^{e}. Now we note the following identities for any test function $w(x,y)$, $F_1(x,y)$ and $F_2(x,y)$ as following

$$\frac{\partial}{\partial x}(w\,F_1)=\frac{\partial w}{\partial x}F_1+w\frac{\partial F_1}{\partial x} \qquad 12$$

Or $$-w\frac{\partial F_1}{\partial x}=\frac{\partial w}{\partial x}F_1-\frac{\partial}{\partial x}(w\,F_1)$$

$$\frac{\partial}{\partial y}(w\,F_2)=\frac{\partial w}{\partial y}F_2+w\frac{\partial F_2}{\partial y} \qquad 13$$

Or $$-w\frac{\partial F_2}{\partial y}=\frac{\partial w}{\partial y}F_2-\frac{\partial}{\partial y}(w\,F_2)$$

Next, we use the component form of the gradient (or divergence) theorem.

$$\iint_{D^e}\frac{\partial}{\partial x}(w\,F_1)\,dA=\int_{\Gamma^e}(w\,F_1)\,n_x\,ds \qquad 14$$

$$\iint_{D^e}\frac{\partial}{\partial y}(w\,F_2)\,dA=\int_{\Gamma^e}(w\,F_2)\,n_y\,ds \qquad 15$$

Where n_x *and* n_y are the components of the unit normal vector such that $\boldsymbol{n}=\langle n_x,n_y\rangle$ on the boundary D^e and ds is the length of an infinitsment line element along the boundary (see figure 3) using Eqs. 12,13,14,15 with

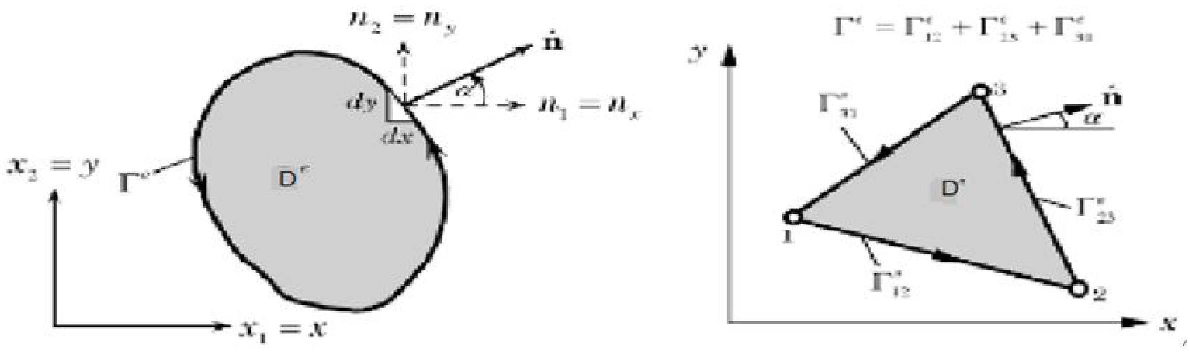

Figure 3 The unit normal vector on the boundary of a finite element

$$F_1 = \frac{\partial u}{\partial x} \qquad , \qquad F_2 = \frac{\partial u}{\partial y}$$

We obtain

$$\iint_{D^e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} - w\, f \right) dA - \int_{\Gamma^e} w \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) ds = 0$$

Or

$$\iint_{D^e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} \right) dA = \iint_{D^e} w\, f \;dA + \int_{\Gamma^e} w\, d_n ds \qquad 16$$

The first integral on the right hand side (RHS) is a line integral taken in the counter-clockwise direction around the boundary Γ^{e}.

Since u is the dependent unknown in the differential equation, we call it the primary variable. The coefficient of the test function w in the boundary term that appeared because of the use of integration by parts.

Here $d_n = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$ is called the secondary variable of the formulation. The specification of the primary variable u on a portion of the boundary is called essential boundary condition and specification of a secondary variable d_n on the boundary is called natural boundary condition.

After we let the weak formulation for our problem, now the function $u(x,y)$ is approximated over a normal element e by

$$u(x,y) \approx U^e(x,y) = \sum_{j=1}^{n} u_j{}^e \, N_j{}^e(x,y) \qquad 17$$

Where $u_j{}^e$ is the value of the function $U^e(x,y)$ at the jth node with coordinates (x_j, y_j) of the element e, and $N_j{}^e(x,y)$ is the interpolation function such that

$$N_i{}^e(x_j, y_j) = \delta_{ij} = \begin{cases} 1 & when \quad i = j \\ 0 & when \quad i \neq j \end{cases} \qquad 18$$

and

$$\sum_{i=1}^{n} N_i{}^e(x,y) = 1 \ , \qquad \frac{\partial N_i{}^e}{\partial x} = \frac{\partial N_i{}^e}{\partial x} = 0 \, .$$

From equation 17 and equation 16 with dropping the superscript e we get

$$\iint_{D^e} \left[\frac{\partial w}{\partial x} \sum_{j=1}^{n} u_j \frac{\partial N_j}{\partial x} + \frac{\partial w}{\partial y} \sum_{j=1}^{n} u_j \frac{\partial N_j}{\partial y} \right] dA = \int_{\Gamma^e} w d_n ds + \iint_{D^e} w \, f \, dA \qquad 19$$

Clearly equation (19) contains n variables $u_1, u_2, \dots\dots, u_n$, This leads us to consider the matrix concepts i.e.,

$$\sum_{i=1}^{n} A_{Ki} \, u_i = b_k \qquad 20$$

where

$$A_{Ki} = \iint_{D^e} \left[\frac{\partial N_k}{\partial x}\frac{\partial N_i}{\partial x} + \frac{\partial N_k}{\partial y}\frac{\partial N_i}{\partial y} \right] dA$$

$$b_k = \int_{\Gamma^e} N_k \, d_n ds + \iint_{D^e} N_k \, f \, dA \qquad 21$$

The linear interpolation function, $N_i(x,y)$, for the triangular element (see figure 4) can be obtained by Grammer's rule as following

$$N_i(x,y) = \frac{D_{j,k}}{C_{i,j,k}}$$

Where

$$D_{j,k} = det\begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \quad , \quad C_{i,j,k} = det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = 2A$$

Hence

$$N_i(x,y) = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \quad , \quad i = 1,2,3 \qquad 22$$

Where A is the area of the triangle and α_i , β_i *and* γ_i are constants known in terms of the nodal coordinate given by

$$\alpha_i = x_j\, y_k - x_k\, y_j \, , \quad \beta_i = y_j - y_k \quad and \quad \gamma_i = -(x_j - x_k).$$

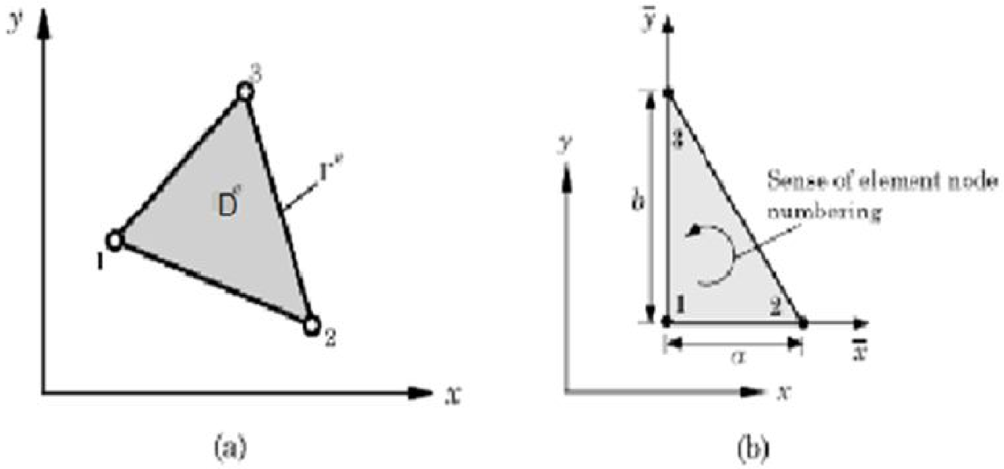

Figure 4 Linear triangular finite element

The values of the element matrices above can be evaluated by numerical integration techniques. Here we used the Gaussian triangle.

3- Assemly and Implementation of Boundary Conditions

We will show the same steps that taken were by Dambaru D. Bhatta's paper. So the assembled equations can be partitioned into:

$$A^{11}U^{1} + A^{12}U^{2} = B^{1} \quad , \quad A^{21}U^{1} + A^{22}U^{2} = B^{2} \qquad 23$$

Where $\{U^1\}$ is the column of known primary variable, $\{U^2\}$ is the column of unknown primary variable, $\{B^1\}$ is the column of unknown secondary variable, and $\{B^2\}$ is the column of known secondary variable. Her we need the continuity of the primary variable along the entire inter-element nodes guarantees the continuity of the primary variables along the entire inter-element boundary. Writing (23) as two matrix equations, by using the operation of matrix we get

$$[A^{11}]\{U^1\} + [A^{12}]\{U^2\} = \{B^1\}$$

and

$$[A^{21}]\{U^1\} + [A^{22}]\{U^2\} = \{B^2\}$$

Now we can see that

$$\{U^2\} = [A^{22}]^{-1}(\{B^2\} - [A^{21}]\{U^1\})$$

By known $\{U^2\}$, then $\{B^1\}$ can be evaluated as

$$[A^{11}]\{U^1\} + [A^{12}]\{[A^{22}]^{-1}(\{B^2\} - [A^{21}]\{U^1\})\} = \{B^1\}$$

Finally, the solution and its derivation are evaluated at any chosen point (x, y) in the element e , so

$$U^e(x,y) = \sum_{j=1}^{3} u_j{}^e \, N_j{}^e(x,y)$$

$$\frac{\partial U^e}{\partial x} = \sum_{j=1}^{3} u_j{}^e \, \frac{\partial N_j{}^e(x,y)}{\partial x} \neq 0$$

$$\frac{\partial U^e}{\partial y} = \sum_{j=1}^{3} u_j{}^e \, \frac{\partial N_j{}^e(x,y)}{\partial y} \neq 0 \qquad 24$$

Note: we stopped because we have used three nodes.

The Intepretation of the Flow Around an Elliiptic Cylinder

The geometric interpretations of the flow, we usually use the stream function ψ in the plane xy. Since the flow is irrotational and symmetric in the direction of x, Therefore, we consider the rectangular shape ABCDE as shown in figure 5.Taken the ψ at the boundaries and no-slip condition i.e., at the curve BC , $\psi = 2$ at ED, $\psi = y$, $\psi = x = 0$ and finally the laplacian $\nabla^2\psi$ zero beside the boundary.

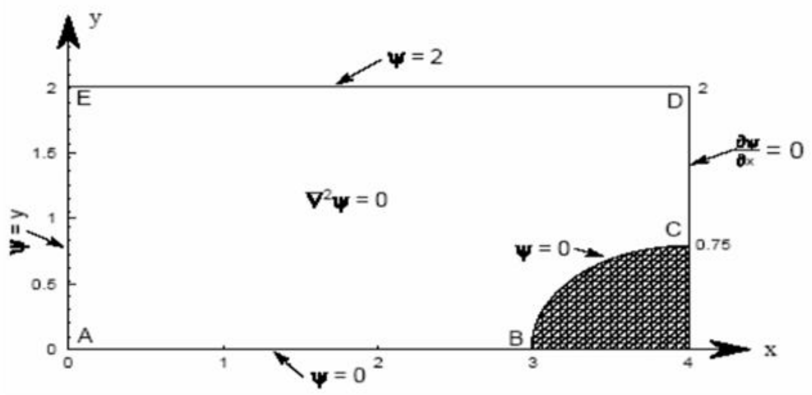

Figure 5 FEM Domain with boundary condition

Conclusion:

What we have introduced in our work is the theoretical point view of the viscous fluid flow around an elliptic cylinder. We have followed almost the same steps that have taken by Dambaru D. Bhatta in his paper. The difference is that we used viscous fluid flow. However, our results remain to be tested using either experimental tolls or computer analysis which we unfortunately do not have. Hence, we based our results on predictions and mathematical analysis. We found that the flow around elliptic cylinder, under the conditions that we have set, is far from the invisid case. This is due to the viscosity element that will have an effect on the surface of the cylinder and as a result it will slow the flow. This remained to be depated and tested to support or deny our results.

As future work we are planning to see what will happen if the fluid flow is magnetized.

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