Using the Linear Quadratic Regulator (LQR) and applying optimal feedback gain matrix (LQR) to improve the system stability

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Abstract:

Linear quadratic (LQ) optimal control can be used to resolve some of the issues, by not specifying exactly where the closed loop eigenvalues should be directly, but instead by specifying some kind of performance objective function to be optimized. Other optimal control objectives, besides the (LQ) type, can also be used to resolve issues of tradeoffs and extra design freedom. We first consider the finite time horizon case for general time varying linear systems, and then proceed to discuss the

infinite time horizon case for Linear Time Invariant systems. Linear Quadratic Regulator (LQR) is an optimal multivariable feedback control approach that minimizes the excursion in state trajectories of a system while requiring minimum controller effort. The behavior of a (LQR) controller is determined by two parameters: state and control weighting matrices. These two matrices are main design parameters to be selected by designer and greatly influence the success of the (LQR) controller synthesis.

1.Introduction.

Linear Quadratic Regulator Technique (LQR) is presented in this project to minimize the cost. Two problems are presented in order to practice computing the state feedback gain matrix K that satisfies the optimal control[1]. The system states are assumed to be controllable & observable, so (LQR) controller can be applied by multiplying the states with matrix k and feed it back again to the system by subtracting them from the input u. The optimized systems are simulated by MATLAB software to ensure the improvement in their responses[2].

2. Theoretical Background:

The system has to be controllable, in order to control it with (LQR). The Q matrix denotes the weights for the system states, and it can be determined from the cost function

$$J(u) = \int_0^{t_f} (x^T Q x + u^T R u) dt + x^T_{(t_f)} Q \, x_{(t_f)} \quad \Rightarrow \quad (1)$$

Or we can select the weights according to our experience with the system by selecting the high weight for the states that have major effect on the required performance and vice versa[2]. Modifying the Q weight with trial and error, which can be done easily by computer program by checking the system response for each try and repeating again until we reach the satisfied performance[3]. The feedback gain matrix K is calculated from the system matrices, Q and R matrices. After applying the feedback to the system the new A matrix for itis computed from Equation (2).

$$A_{LQR} = A - BK \quad \Rrightarrow \quad (2)$$

The block diagram for the system with the optimal feedback gain matrix (K) [4] is shown in figure 1 and the block diagram of a basic feedback control loop is shown in figure 2.

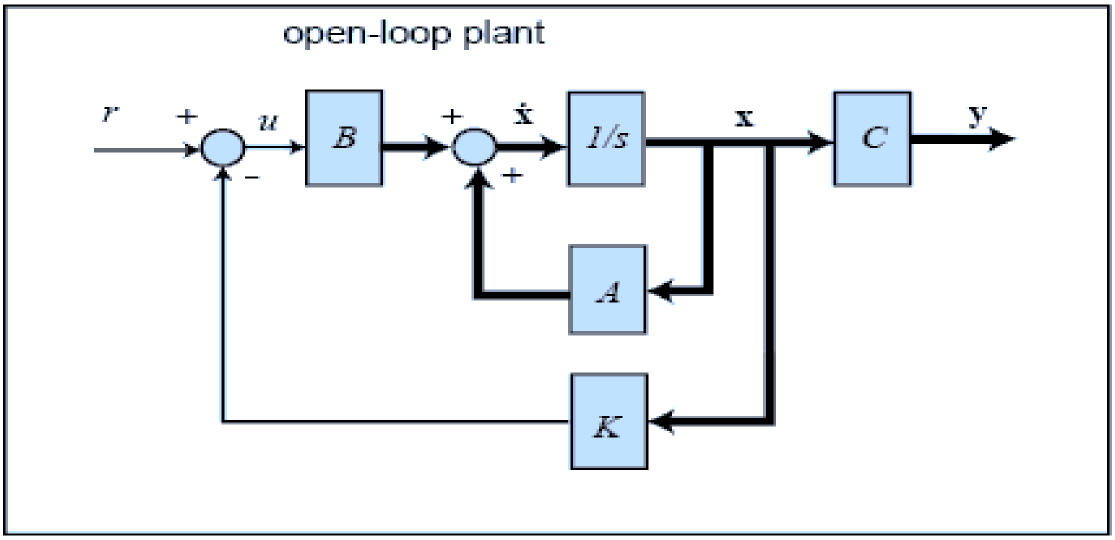

Figure 1. The block diagram for the system with the optimal feedback matrix

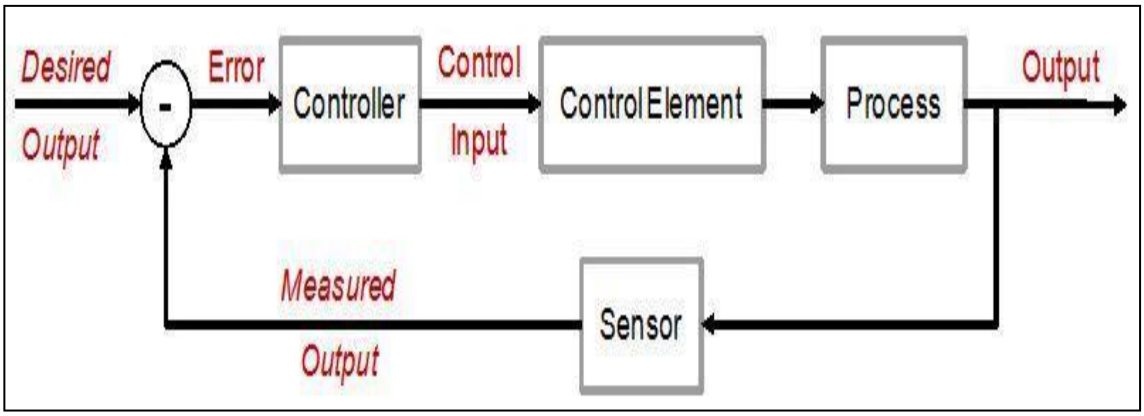

Figure 2. The block diagram of a basic feedback control loop

3. Problems Description:

3.1 For the system:

$$\dot{x}_1 = x_2 - x_1 \quad \Rrightarrow \quad (3)$$

$$\dot{x}_2 = -2x_1 - 3x_2 \quad \Rrightarrow \quad (4)$$

LQR controller has to be designed and implemented to minimize

$$J(x,u) = \int_{t_0}^{t_f} \frac{1}{2}[x_1^2(t) + x_2^2(t) + u^2]dt \quad \Rrightarrow \quad (5)$$

At $t_0 = 0,\ t_f = 10$ time units, and $x_{(0)} = [4,2]^T$

A) First, P(t) matrix has been solved numerically Using the RDE. a plot of the three distinct elements of P(t) versus time is shown. For both methods we are starting at the final condition, and then propagating it backwards in time.

B) Second, the matrix $P(t_0)$ was used to obtain a steady-state approximation of the time-varying LQR. The steady-state LQR was implemented by using it to close the loop. The state vector $x^*(t)$ was simulated and it converged to zero and optimal control $u^*(t)$ was plotted

C) The MATLAB command `are' was used to compute the matrix P corresponding to the infinite-horizon problem, and verify that it is approximately the same as steady-state solution from part (b).

3.2 For the linear system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + Bu \quad \Rrightarrow \quad (6)$$

Consider each of the following cases. For each case, verify whether all the assumptions of the LQR theorem are satisfied, and investigate it via simulation (let $t_f = 5$ what are the consequences in each case)[5]. Use the code from Problem 1 to compute the matrix P(t) and approximate its steady-state value, and then to compute the optimal controller[6]. In each case below, report the LQR gain, show plots of P(t), x(t) and u(t), and also compute the optimal cost corresponding to the infinite-horizon case.

a. Let $B = [1,1]^T, Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \ R = 0.5$

b. Let $B = [1,1]^T, Q = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \ R = 0.5$

c. Let $B = [0,1]^T, Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \ R = 0.5$

d. Let $B = [1,1]^T, Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ R = 0.5$

We will using the MATLAB function `cholp' to factor $Q = C^T C$ when Q is positive-semi definite.

For the state equations are:

$$\dot{x_1} = x_2 - x_1 \quad \Rrightarrow \quad (7)$$
$$\dot{x_2} = -2x_1 - 3x_2 + u \quad \Rrightarrow \quad (8)$$

And the cost function which need to be minimized:

$$J(u) = \int_{t0}^{tf} 1/2[x_1^2 + x_2^2 + u^2]dt \quad \Rightarrow (9)$$

Where$t_0 = 0 \; and \; t_f = 10$, and the initial boundaries are $x(0) = [4{,}2]^T$

From this system we obtain:

$$A = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow (10)$$

From system's cost function, also we obtain:

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \; and \;\; R = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \;\; \Rightarrow \;\; R^{-1} = 2 \;\; \Rightarrow (11)$$

Moreover, to check the controllability we need to find rank of the matrix $C_n = [\,B \; AB]$

$$C_n = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \;\; \Rightarrow \;(12)$$

Which must be of rank 2 to get a fully controllability system[7], when checking C_n is indeed has rank 2

Which means the same length of A, then the system is fully controllable.

Then, to check the observability property between system matrix A and the matrix C, we will need to find the rank of the matrix$O_b = [C \; AC]^T$. However, [7]we will need to find the matrix C by decompose the Q matrix by MATLAB function *cholp(Q)*

$$C = \begin{bmatrix} 0.707 & 0 \\ 0 & 0.707 \end{bmatrix} \Rightarrow (13)$$

Now, we can check for the observability property matrix through:

$$O_b = \begin{bmatrix} 0.7071 & 0 \\ 0 & 0.7071 \\ 0.7071 & 0 \\ 0 & -1.4142 \end{bmatrix} \Rightarrow (14)$$

Since, the rank of O_b matrix is 2, and then the system is fully observable. Also, the system is controllable and observable and the LQR can achieve the optimal close-loop control law $u^*(t) = -kx^*(t)$ where k is the LQR gain and given by $K = R^{-1}B^TP$

The solution of RDE:

The analytic matrices of $P^{\cdot}(t)$,using Reccatti Differential Equation is given by:

$$P^{\cdot}(t) = -AP - A^TP - Q + PBR^{-1}B^TP \Rightarrow (15)$$

Where P is a matrix on the form:

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \Rightarrow (16)$$

By applying the give matrices in RDE, we have:

$$P^{\cdot}(t) = \begin{bmatrix} 2P_{12}^2 + 4P_{12} + 2P_{11} - 0.5 & 4P_{12} - P_{11} + 2P_{22} + 2P_{12}P_{22} \\ 4P_{12} - P_{11} + 2P_{22} + 2P_{12}P_{22} & 2P_{22}^2 + 6P_{22} - 2P_{12} - 0.5 \end{bmatrix} \Rightarrow (17)$$

Then, we have

$$\dot{P}_{11} = 2P_{12}^2 + 4P_{12} + 2P_{11} - 0.5$$

$$\dot{P}_{12} = 4P_{12} - P_{11} + 2P_{22} + 2P_{12}P_{22} \Rightarrow (18)$$

$$\dot{P}_{22} = 2P_{22}^2 + 6P_{22} - 2P_{12} - 0.5$$

These Equations have been solved by MATLAB and to find boundary conditions $P(t_f) = 0$ and it propagates backward in time for 10 time unit with transient state near and steady state after a while of propagating to stop at zero with a constant value.

Also, we can find that:

$$P_{(t)} = \begin{bmatrix} P11_{(t)} & P12_{(t)} \\ P21_{(t)} & P22_{(t)} \end{bmatrix} \Rightarrow (19)$$

$P12_{(t)} = P21_{(t)}$ Because $P_{(t)}$ is symmetrical matrix

At $t = 0$, we obtain:

$$P_{(0)} = \begin{bmatrix} P_{11(0)} & P_{12(0)} \\ P_{12(0)} & P_{22(0)} \end{bmatrix} \Rightarrow (20)$$

By computing $P_{(t)}$ at $t = 0$ with matlab, we got

$$P_{(0)} = \begin{bmatrix} 0.2241 & 0.0129 \\ 0.0129 & 0.0852 \end{bmatrix} \Rightarrow (21)$$

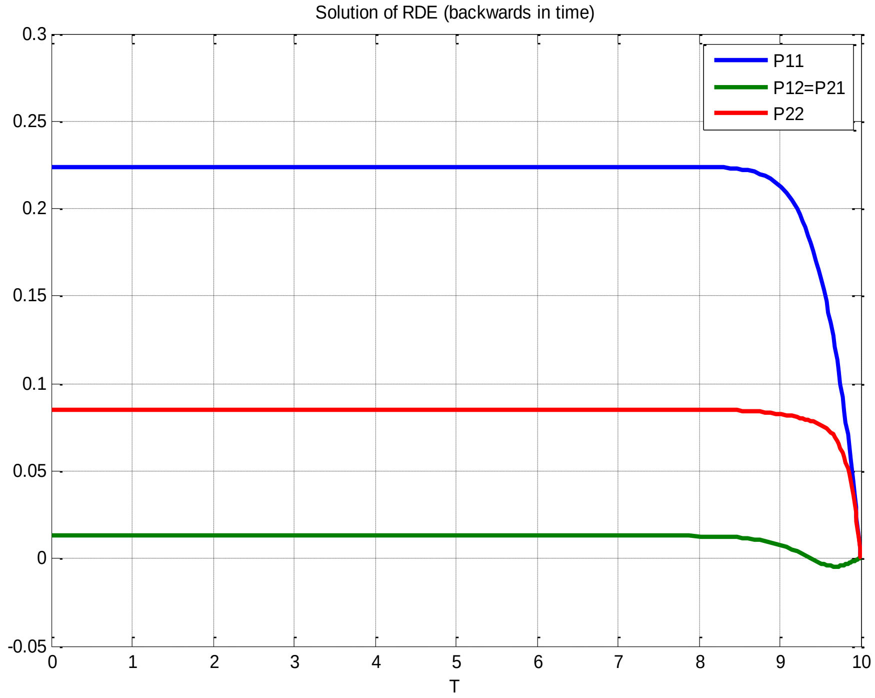

Figure 3. Solution of RDE (back ward in time)

Finding $x^*(t)$ and $u^*(t)$ $from$ $P(0)$. We need to take value of $P(t)$ at $t = 0$ to obtain the steady state approximation of the time-varying LQR. And, we can find the LQR gain vector $K(1xn)$ through.

$$K = R^{-1}B'P(0) \Rightarrow (21)$$

$$K = 2 * [0\ 1] \begin{bmatrix} 0.2241 & 0.0129 \\ 0.0129 & 0.0852 \end{bmatrix} \Rightarrow (22)$$

Then, we obtain $K = [\,0.0257\ 0.1704] \Rightarrow (23)$

The (LQR) gain will be used to find the close-loop (LQR) control law through $u^*(t) = -Kx$ which starts evaluating by the initial conditions of $x(0) = [4\ 2]^T$. the curve of $u(t)$ is shown in figures 4 and 5.

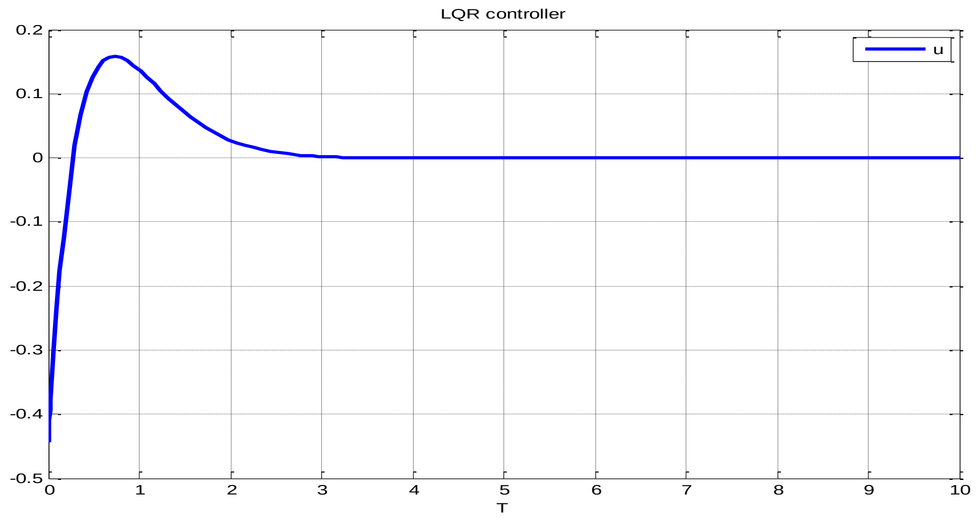

Figure 4. The curve of $u(t)^*$

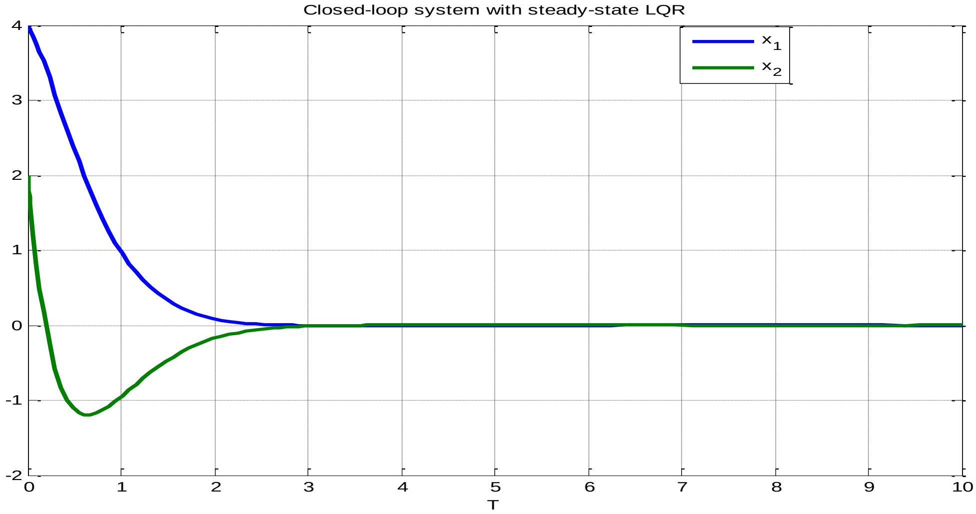

Figure 5. The curve of $x(t)^*$

Finding ARE (infinite - Horizon problem). In this part of problem, the first derivative turns to be zero by Kalman theorem. Also, the differential equation is converted to an algebraic equation.

By using MATLAB command "$are(A, B, Q)$" where B is $BR^{-1}B^T$, we can find directly the steady state approximation of differential Riccati equation which is the same DRE evaluated at $t = 0$

$$P = \begin{bmatrix} 0.2241 & 0.0129 \\ 0.0129 & 0.0852 \end{bmatrix} \Rightarrow (24)$$

3.3 For the linear system:

$$x^{\cdot} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + Bu \Rightarrow (25)$$

With $t_f = 5$ and $x(0) = [4\ ,\ 2]^T$ and $P(t_f) = 0 \Rightarrow (26)$

$$J(u) = \int_{t0}^{tf} 1/2[2x_1^2 + x_2^2 + u^2]dt \Rightarrow (27)$$

We have from the system:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad R = \frac{1}{2} \Rrightarrow R^{-1} = 2 \Rrightarrow (28)$$

As shown in figure (1.1), just for illustration of another way of implementing a linear time-invariant system in MATLAB and to give a clue for the reader how the MATLAB code works. Check for (LQR) Theorem for each case:

- **Case (A):**

$$B = [1,1]^T, \; Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \Rrightarrow (29)$$

The system is completely controllable if the controllability matrix C_n has a full rank (i.e. rank ($C_n = n$), where n is the dimension of matrix A and $C_n = [B \;\; AB \;\;\; A^2B \ldots\ldots A^{n-1}B]$

$$C_n = [B \;\; AB\;] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \Rrightarrow (30)$$

Since $\rho(C_n) = 2$, the system is completely controllable

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \Rrightarrow (31)$$

then, the eigen value (Q) = 0.5, 0.5 and Q is P.D, so the system is fully controllable and Q matrix is P.D, Also we can apply (LQR) theorem on this particular system. The computed (LQR) gain is

$$P = \begin{bmatrix} 1.3568 & -0.0892 \\ -0.0892 & 0.1244 \end{bmatrix} , K = [\,2.5351 \quad 0.0704] \; , \; C = \begin{bmatrix} 0.7071 & 0 \\ 0 & 0.7071 \end{bmatrix} \Rrightarrow (32)$$

The optimal cost corresponding to the infinite-Horizon is $J(x_0) = x_0^T P(0) x_0 \;\Rrightarrow (33)$

$$J(x_0) = 20.7787$$

The resulting curves illustrate The applying LQR theorem on the system $x^*(t) and\ u^*(t)$ respectively.

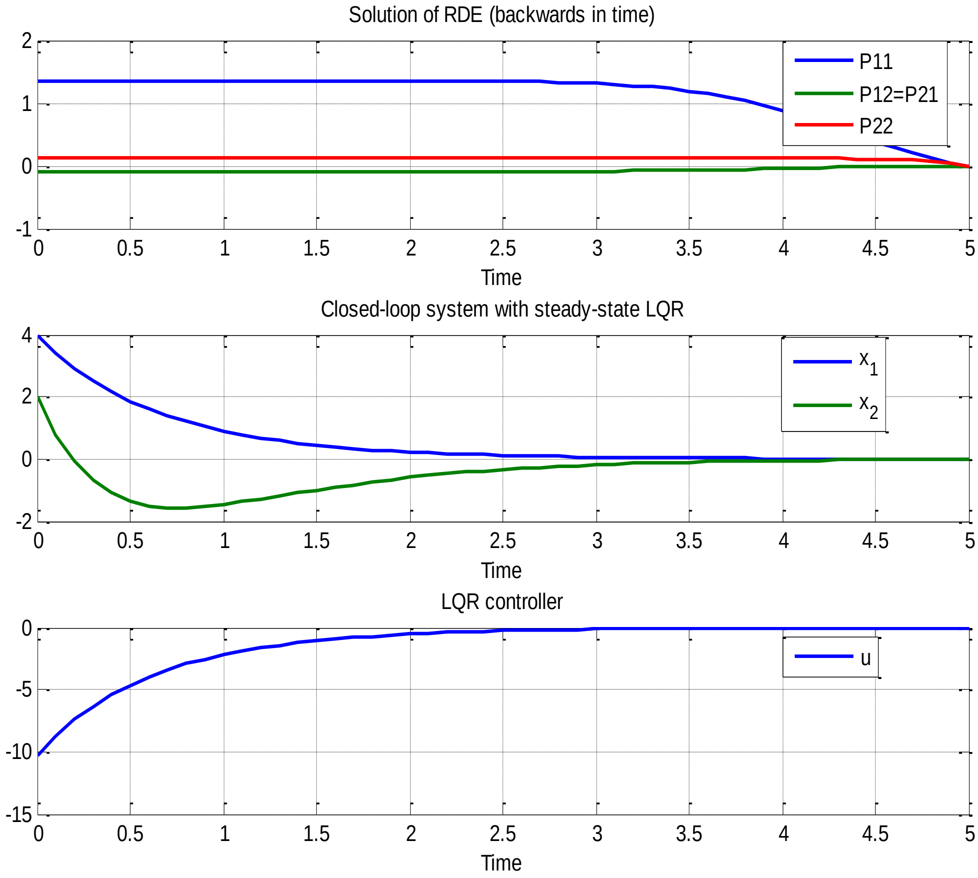

Figure 6. The curve of Case (A)

- **Case (B):**

$$B = [1,1]^T, \qquad Q = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow (34)$$

The controllability $C_n = [B \ \ AB \ \ A^2B \ldots\ldots A^{n-1}B]$ of the system with these matrices

$$C_n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \Rrightarrow (35)$$

The rank of $(C_n) = 2$, then the system is fully controllable and we need to check Q

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \Rrightarrow (36)$$

The eigen value (Q) = 0.5 and Q is P.S.D. Therefore, we need to apply another property to check if LQR satisfies. By decomposing Q by Cholesky decomposition property $Q = C^T C \ \Rrightarrow C = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$,and then check the observability test between A and C.

$$O_b = \begin{bmatrix} 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \Rrightarrow (37)$$

O_b Matrix has rank of 2, and the theorem of (LQR) can achieve the optimal controller we are looking for. The computed LQR gain is

$$P = \begin{bmatrix} 1.5978 & 0.2989 \\ 0.2989 & 0.1494 \end{bmatrix}, K = [\,3.7933 \quad 0.8967], C = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rrightarrow (38)$$

The system is completely controllable and completely observable and, the optimal cost corresponding to the infinite-Horizon is $J(x_0) = x_0^T P(0) x_0 \ \Rrightarrow (39)$

$$J(x_0) = 30.9441$$

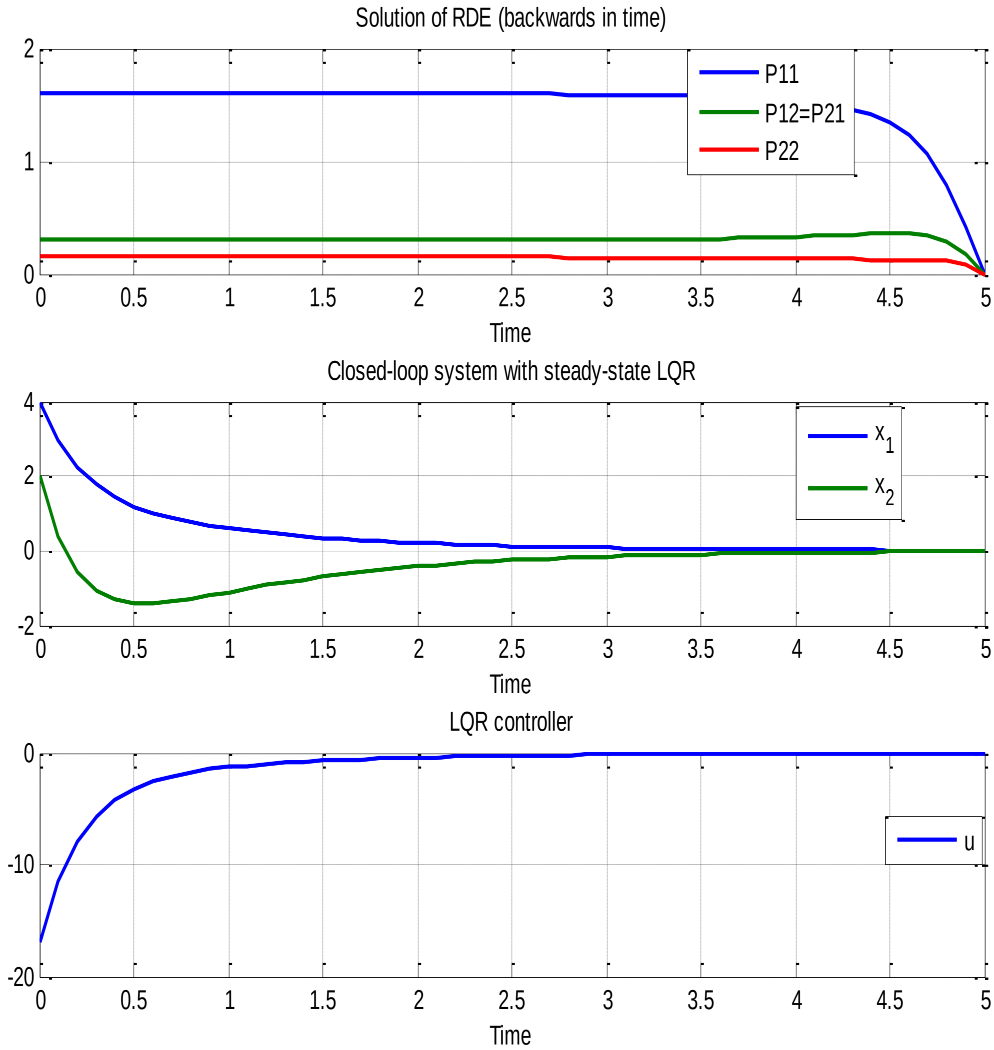

Figure 7. The curve of Case (B)

- **Case (C):**

$$B = [0,1]^T,\ Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \Rightarrow (40)$$

In this case, as noticed the system states are decomposed and by taking $B = [01]^T$, the controller will has no effect on state x_1 That means system will never be controlled by LQR controller. In spite of the fact Q is P.D, and these system parameters have been manipulated on the same code, and the resulted curves were as expected, the input fails to stabilize state x_1; as shown in Figure (1.8).

LQR gain is:

$$P = \begin{bmatrix} 5507.2 & 0 \\ 0 & 0.1180 \end{bmatrix} , K = [0 \quad 0.2361] , C = \begin{bmatrix} 0.7071 & 0 \\ 0 & 0.7071 \end{bmatrix} \Rightarrow (41)$$

The system is not completely controllable but it is completely observable, and the optimal cost corresponding to the infinite-horizon is $J(x_0) = x_0^T P(0) x_0 \Rightarrow (42)$

$$J(x_0) = 8.8115e + 04$$

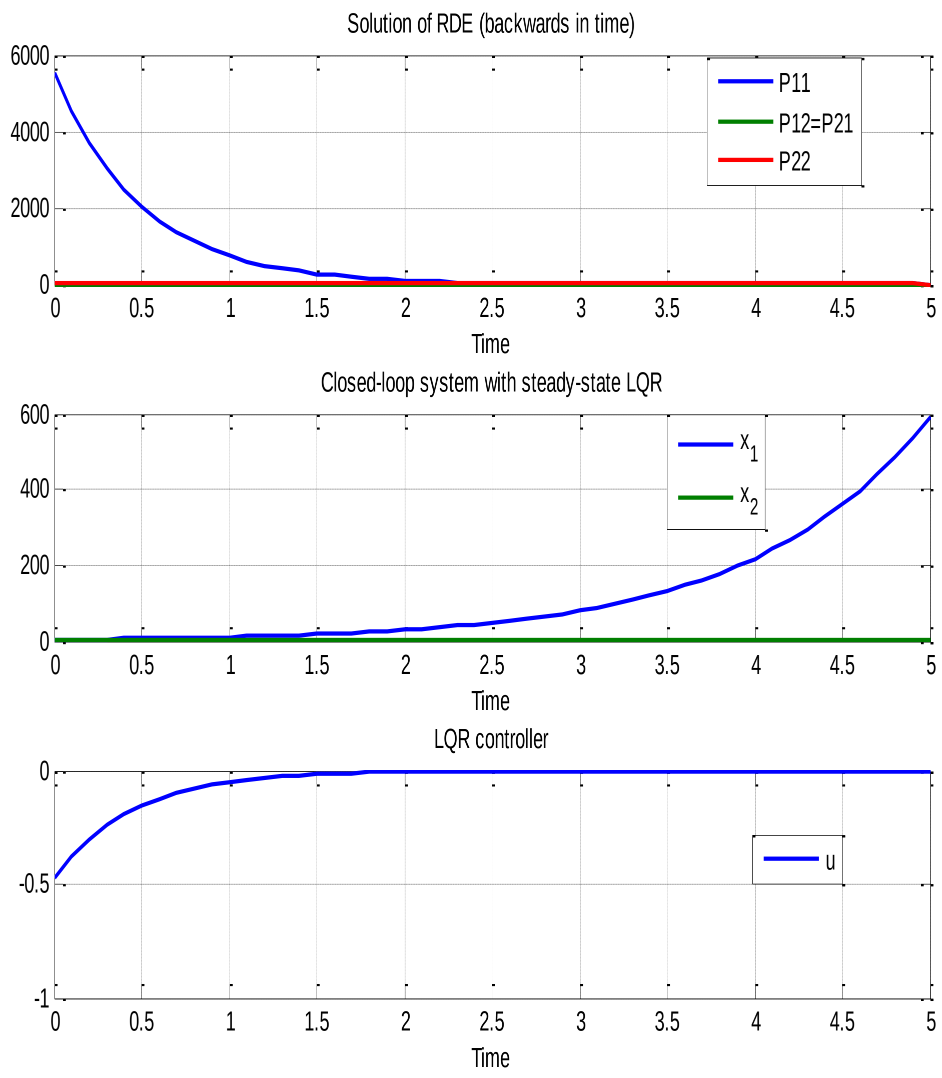

Figure 8. The curve of Case (C)

- **Case (D):**

$$B = [1,1]^T,\ Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rrightarrow (43)$$

In this case, directly the system is fully controllable. We need only to check the matrix Q which also, clear it is a P.S.D from the diagonal eigen value 0, 1. We need to decompose it according to Cholseky decomposition. We have $Q = C^T C \ \ and \ \ C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Additionally, we need to check the last condition of main (LQR) theorem, which is (A,C) which must have the observability property, then (LQR) theorem can satisfy.

$$O_b = [C \ \ AC\ ]^T$$

$$O_b = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \Rrightarrow (44)$$

And, this matrix has rank of 1 and then The LQR theorem can achieve controller for this case of system. Figure (1.9) proves this statement. The computed (LQR) gain is

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0.2247 \end{bmatrix} ,K = [\,0 \quad 0.4495\,] ,C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rrightarrow (45)$$

The system is completely controllable but it is not completely observable, and the optimal cost corresponding to the infinite-horizon is $J(x_0) = x_0^T P(0) x_0 \implies J(x_0) = 0.8990 \Rrightarrow (46)$

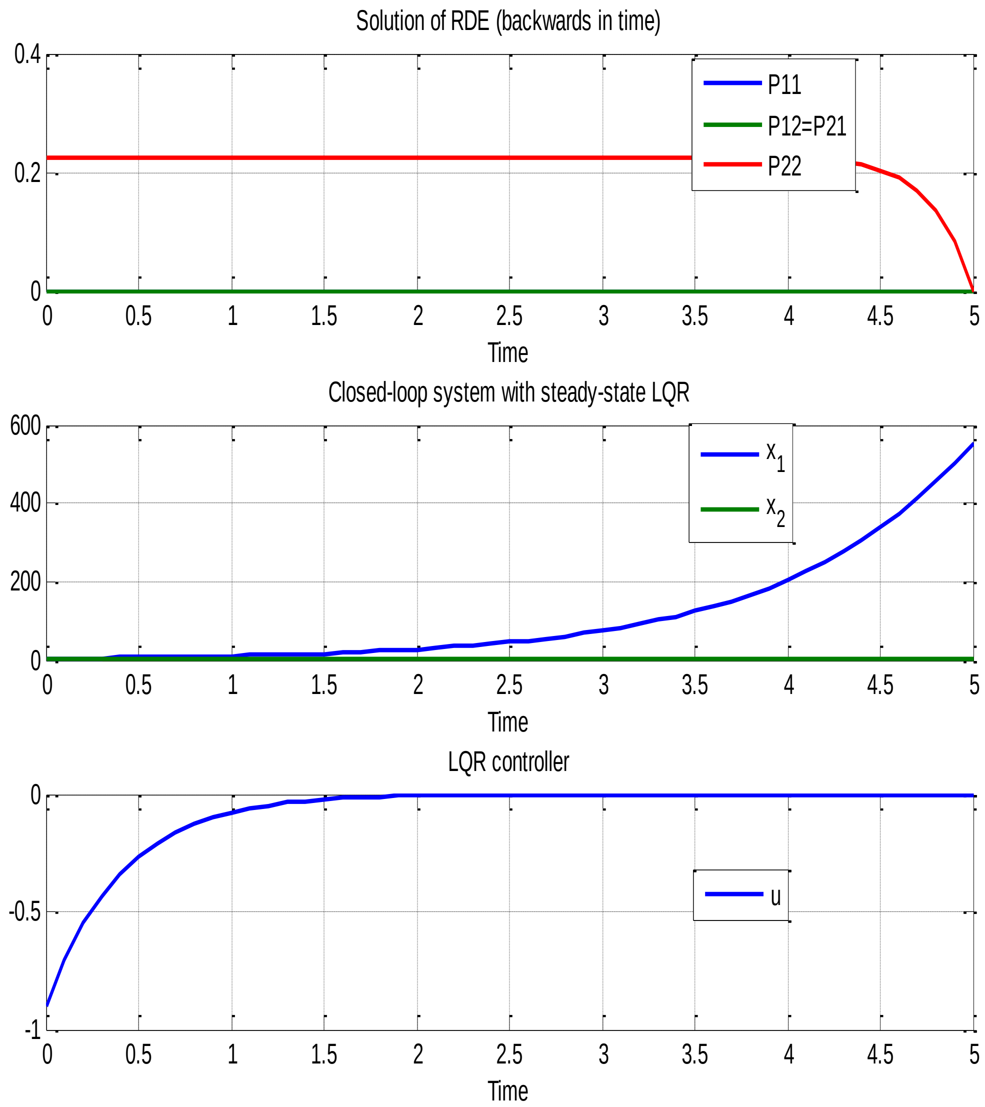

Figure 9. The curve of Case (D)

Now, we can see the summary in table 1:

Table 1: The properties for each system cases

Cases	Case(A)	Case(B)	Case(C)	Case(D)
Is the system controllable	Yes	Yes	No	Yes
Is the system observable	Yes	Yes	Yes	No
Is Q Semi-positive-definite (S.P.D) or positive-definite (P.D)	P.D	S.P.D	P.D	S.P.D
Is the system stable	Yes	Yes	No	No

Table 1 shows the properties for all cases for the system. According to the results in case (A) and case (B) we can conclude that the optimal control (LQR) can be applied to the system if it is controllable, observable and Q is P.D or S.P.D. From case (C) and case (D) if the system is not controllable or not observable then LQR can't be applied.

4. Conclusion & Recommendation:

The system has to be controllable and observable, so we can apply optimal feedback gain matrix (LQR) to improve the system stability. (LQR) can be designed to improve certain control criteria this depends on the weight selection for Q and R. To conclude a general idea in this problem, we need to summarize all the previous case. In case (A), the system was controllable and the Q matrix a P.D, so the LQR is satisfied. As it has been seen, the controller succeeded to stabilize the states with cost of 20.7787.

In case (B), Q matrix has been chosen with much higher weights for state x1 slightly high for x_2. As a result, states have convergence to zero is much quicker. On the other hand, the cost is 30.9441. It is higher by about 50% of cost of case(A). So, we are in compremise situation between time and cost in choosing Q matrix. In case (C), since the system had decomposed state and the matrix $B = [0\ 1]^T$, the LQR theorem was useless for this case system as the curves have shown .In the last case, the only obstacle using (LQR) theorem was that observability property between A and C matrices has failed.

5. References:

[1] Akay B. (2013). A study on particle swarm optimization and artificial bee colony algorithms for multilevel thresholding. Appl. Soft. Comput. 13, 3066–3091.

[2] Ali D., Hend L., Hassani M. (2008). Optimised Eigenstructure assignment by ant system and LQR approaches. Int. J. Comput. Sci. Appl., 5, 45-56.

[3] Amini F., Hazaveh N.K., Rad A.A. (2013). Wavelet PSO-based LQR algorithm for optimal structural control using active tuned mass dampers. Computer-Aided Civil and Infrastru. Eng., 28, 542–557.

[4] Chen R., Sung W. (2012). A PSO-based LQR controller for accelerator PWM power supply. Adv. Mat. Res., 490, 71-75.

[5] Eberhart R., Kennedy J. (1995). A New Optimizer Using Particle Swarm Theory, "Proc. 6th Symp. Micro Mach. Man. Sci.", Nagoya, Japan, Oct. 4-6, pp. 29–43.

[6] Ghoreishi S.A, Nekoui M.A. (2012). Optimal weighting matrices design for LQR controller based on genetic algorithm and PSO. Adv. Mat. Res., 443, 7546-7553.

[7] Guoping L., Genfu X., Xiaohui Y. (2010). The LQR Real-time Control for the Inverted Pendulum Based on PSO, "Int. Conf. Elect. And Contr. Eng. ", Wuhan, China, June 25-27, pp. 2363-2366.