The Divisors Graphs of Finite Commutative Rings

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Abestract:

Let R be a finite commutative ring. Define a graph $G(R)$ with vertices in R such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u,v) = 1$. Such graph will be called divisor graph. This study will focus on investigating the degrees of vertices, radius, diameter, center, cycles, peripheries, clique and some of coloring properties for the highlighted graph. We also proof some interesting theorems related to these concepts and calculate the complete subgraphs of $G(R)$. Finally, we supported our study with figures of certain graphs and their complements.

Keywords: *Finite Ring, Divisor Graph, coloring, degree of vertex,* cycles.

I. Introduction:

Associating a graph with an algebraic structure is a research interest subject and has drawn extensive attention. Often, researches in this subject aim at exposing the relationship between algebra and graph theory and at advancing applications of one to the other. The idea to associate a graph to a commutative ring, where all elements of the ring are vertices of this graph, first appeared in [1], which deals with graph coloring. The notion of a divisor graph was early investigated in [2] and also studied further in [3]. Over the past two decades, significant number of researchers have been worked on zero-divisor graphs, and many similar studies publicized as those within references[4,5], for instance. One can associate other graphs to rings or other algebraic structures. The benefit of studying such graphs is that one may find some results about the algebraic structures and vice versa. In fact, there are three major problems in this area: Description of the resulting graphs, description of the algebraic structures with isomorphic graphs, and realization of the connections between the structures and the corresponding graphs. This is a particularly interesting subject for some graph theorists and algebraists, since it relates two very different areas of mathematics, involves exciting computations, and can be studied at many different levels of mathematical expertise and sophistication.

In this article, the definition we will provide for the divisor graph of a ring takes quit different manner and has not been seen in the literature before. By this study we hope to enrich the graph-theoretic properties of the g.c.d. graph, which will help in, better understand the ring-theoretic properties of R. The tools we used to aid in the research process in this study are Mathematica notebooks that displayed the divisor graph of certain rings. All of the graphs displayed in this paper were generated using those notebooks to help in studying larger divisor graphs.

Specifically, our study offers the chance to study the interplay between the ring theoretic properties of finite commutative ring of integers $R = \mathbb{Z}_n$ and the theoretic properties of the related divisor graph $G(R)$. We study clique (complete subgraphs) of $G(R)$ and give an explicit formula of cliques in $G(R)$. The degree of vertices as well as the maximum degree $\Delta(G)$ and the minimum degree $\delta(G)$.

In section II, we will provide basic definitions and notations in graph and number theory that used throughout this paper. Basic reference for graph theory one can refer to is [6].

Section III, contains the main results of this work that presented by stating and proving theorems introduced within this section. Finally, we conclude in Section IV.

II. Basic concepts:

Throughout this section, we summarize some well-known basic facts of $G(R)$. First, we recall that in a connected graph G, the eccentricity $ecc(v)$ of a vertex v is the distance between v and a vertex farthest from v in G. Hence, referring to the eccentricity, we can define the radius $rad(G)$ of the graph G as the smallest eccentricity among the vertices of G. The distance between any two vertices in a graph G is the number of edges in a shortest path connecting them. This is also called the geodesic distance. The diameter of G is the greatest distance between any two vertices of G. From the eccentricity point of view, the definition of the center of a graph G is the set of vertices v, such that $ecc(v) = rad(G)$. The periphery of G is the set of vertices v, such that $ecc(v) = diam(G)$. Recall that the length of a smallest cycle in a graph G is the girth of G, denoted by $gr(G)$. The graph G is called complete if every pair of vertices of G is adjacent, i.e., every two vertices share exactly one edge.

A clique C, in an undirected graph G is a subset of the vertices C, such that every two distinct vertices are adjacent. This is equivalent to the condition that the subgraph of G induced by C is complete. The clique number $\omega(G)$ of a graph G is the order of the largest clique (complete subgraph) of G.

Graph coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. This is also known as vertex coloring. The minimum number k for which a graph G is k-colorable is called its chromatic number, and denoted $\chi(G)$. If $\chi(G) = k$, then the graph G is said to be k-chromatic.

The degree of a vertex of a graph G is the number of edges incident to the vertex, with loops counted twice. The degree of a vertex v is denoted $deg(v)$. The maximum degree of a graph G, denoted by $\Delta(G)$, and the minimum degree of a graph, denoted by $\delta(G)$, are respectively, the maximum and minimum degree of its vertices. It is clear in a regular graph; all degrees are the same.

Note that, a vertex with a loop "sees" itself as an adjacent vertex from both ends of the edge thus adding two, not one, to the degree.

The edge connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G. However, the vertex connectivity $\mathrm{K}(G)$ of a connected graph G (other than a complete graph) is the minimum number of vertices whose removal disconnects G.

In number theory**,** the prime counting function, is the function $\pi(x)$ giving the number of primes less than or equal to a given number x.

We introduce the following proposition, which states some important properties of prime numbers that related to our negotiation.

Lemma 1: *Let p be a prime and $p|a_1a_2 \dots a_n$, where $a_1, a_2, \dots, a_n$ are positive integers, then $p|a_i$ for some i, where $1 \leq i \leq n$.*

Proof: see Ref. [7].

Corollary 1: *If $p, q_1, q_2, \dots, q_n$ are primes such that $p|q_1q_2 \dots q_n$, then $p = q_i$ for some i, where $1 \leq i \leq n$.*

Proof: see Ref. [7].

Theorem 1: (The Fundamental Theorem of Arithmetic) *Every integer* $n \geq 2$ *either is a prime or can be expressed as a product of primes. The factorization into primes is unique except for the order of the factors.*

Proof: see Ref. [7].

It follows from this theorem that every composite number n can be factorized into primes. Such a factorization is called a prime factorization of n.

Corollary2. *Let* $a, b \in \mathbb{Z}_n$. *If* $(a, b) = 1$ *and* n *is a natural number, then* $(a^n, b^n) = 1$.

Proof: see Ref. [7].

III. Main results:

In this section, we study the properties of the graph referring to the definitions enclosed in the previous section.

Let $A = \{0, 1, 2, \dots, n-1\}$, we define a mapping $\varphi: \mathbb{Z}_n \to A$, by $\varphi([a]_n) = a$. It is clear that φ is bijective. From now on we denote elements of A by $v_i = i$ for all i in A. So, $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$.

Theorem 2:

Let $G = G(\mathbb{Z}_n)$. *Then:*

1- $dim(G) = 2$

2- $rad(G) = 1$.

3- $C(G) = \{v_1\}$.

4- $Preiph(G) = V(G) \setminus \{v_1\}$.

Proof: Let $G = G(\mathbb{Z}_n)$ be a g.c.d. graph, and let $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$, then two cases for the eccentricity of v_i can be determined:

Case (I): If $i = 1$, then g.c.d. $(v_1, v_j) = 1$ for all j which means $ecc(v_1) = 1$.

Case (II): If $i \neq 1$, then for any $j \neq 1$, the shortest path from v_i to v_j is $v_i v_1 v_j$. This yields $d(v_i, v_j) = d(v_i, v_1) + d(v_1, v_j) = 1 + 1 = 2$. Therefore, $ecc(v_i) = 2$.

From *Case(I)* and *Case(II)*above, one concludes respectively:

1-$ecc(v_1) = 1 = rad(G)$,this implies $C(G) = \{v_1\}$,

2-$ecc(v_i) = 2 = dim(G)$, for all $i \neq 1$, this implies$Preiph(G) = \{v_i : i \neq 1\}$.■

Corollary 3:

If G is a g.c.d. graph, then:

1-$V(G) = C(G) \cup Preiph(G)$.

2-The vertex v_1 is adjacent to every other vertex in G.

3-G is connected.

4-$gr(G) = 1$.

5-$G$is not simple. This comes from the fact that $gr(G) = 1$.

6-$G$is not a complete graph, because v_0 is not adjacent to any v_i, $i \neq 1$. See Figs.1-4.

Note that, the complement of the graph G is not connected, not simple and not complete see figs. 5-8.

Theorem 3: *The set number of primes union $\{1\}$ is the clique.*

Proof: Let $S = \{v_i \epsilon V: i\ is\ a\ prime\} \cup \{1\}$, that means $g.c.d.(v_i, v_1) = 1$ and $g.c.d.(v_i, v_j) = 1$ for all $v_i, v_j \epsilon S$. Hence S is the clique set in G.■

Theorem 4: *The clique number* $\omega(G) = \pi(n-1) + 1$**.**

Proof: To prove $\omega(G) = \pi(n-1) + 1$ we must prove that the set S is the largest clique set. Let $S_1 = S \cup \{w\}$, where w is not in S, then there exist $v \epsilon S$ such that $g.c.d.(v, w) \neq 1$, this implies that v is not adjacent to w; which means S_1 is not clique set, so S is the largest clique set. Hence, $\omega(G) = |S| = \pi(n-1) + 1$.■

Theorem 5: *The chromatic number* $\chi(G) = \pi(n-1) + 1$.

Proof: Let $S = \{v_i \epsilon V : i\ is\ a\ prime\} \cup \{1\}$ then for all v_i, v_j in S, $g.c.d.(v_i, v_j) = 1$. This means every element of S will have a different color. Now if e is a vertex in G such that $e \notin S$ then by theorem 1, e can be expressed as a product of primes, then e will take the same colour of one of its prime factors, so the minimum number for which a graph G is colourable equals $|S|$. Hence, $\chi(G) = \pi(n-1) + 1$. ■

Proposition1: *In this graph* $\lambda(G) = K(G) = 1$.

Proof**:** The minimum number of edges which make G is disconnected is one; that is the edge $e = \{0,1\}$. In the same manner, the minimum number of vertices which make G is disconnected is one; that is the vertex $v_1 = 1$.■

Proposition 2: *The longest cycle of* $G(\mathbb{Z}_n) = n-1$, *and The shortest cycle of* $G(\mathbb{Z}_n) = 1$.

Proof**:** Since $g.c.d(v_i, v_{i+1}) = 1$ for all $v_i, v_{i+1} \in V(G)$ except the vertex v_0**,** then we have a cycle of length $n-1$. If we suppose that there is a cycle of length greater than $n-1$, then according to the definition of cycle, we need at least one more vertex, which is impossible. ■

Corollary 4: *There are no cycles of length* 2.

In this graph, the degree of any vertex is presented in the following theorem:

Theorem 6:

1) The *maximum degree* $\Delta(G) = n + 1$ *at the vertex* v_1, *and the minimum degree* $\delta(G) = 1$ *at the vertex* v_0,

2) If v_i *is a* prime *such that* $mv_i > n$ *for all positive integers* $1 < m < n$, *then* $deg(v_i) = n - 2$,

3) If v_i *is a* prime *such that* $kv_i < n$ *for some positive integers* $1 < k < n$, *then* $deg(v_i) = n - 2 - |B|$, *where* $B = \{kv_i < n : k \in \mathbb{N}\}$.

Proof: (1) We know that $g.c.d.(v_1, v_j) = 1$, for all $v_j \in \mathbb{Z}_n$. Since $g.c.d.(v_1, v_1) = 1$, then $deg(1) = n + 1$. Similarly, the property $g.c.d.(v_0, v_j) = 1$ holds only when $v_j = 1$. Thus, $deg(v_0) = 1$.

(2) Suppose that v_i is a prime such that $mv_i > n$ for all positive integers $1 < m < n$ then v_i is adjacent to all numbers except v_0 and v_i itself. Thus, $deg(p) = n - 2$.

(3) Suppose that v_i is a prime such that $kv_i < n$ for some positive integers $1 < k < n$ then v_i is adjacent to all primes and numbers, which p doesn't appear in its factorization. In addition, it is not adjacent to v_0 and v_i itself. Thus, $deg(p) = n - 2. -|B|$.■

Let $v_i \in A$ be any non-prime, then v_i can be factorized into primes. Therefore, v_i is not adjacent to v_0, and v_i itself, and any vertex that has one factor of v_i in its factorization, which means

$$deg(v_i) = n - 2 - \#(vertices\ which\ have\ one\ factor\ of\ v_i\ in\ its\ factorization)$$

See Figs. 1-4 below.

Remark: *G is not a regular graph, because $\delta(v_0) \neq \Delta(v_1)$ in $G(\mathbb{Z}_n)$, $n > 2$. See* Figs.1- 4.

Underneath, we introduce some calculations of the clique number for the graphs corresponding to the finite rings $\mathbb{Z}_\mathrm{n}$ and $n = \{2,3, \dots ,98\}$.

Table 1. Clique number for the graphs corresponding to finite rings $\mathbb{Z}_\mathrm{n}$, for $n = 2, 3, \dots, 98$.

n	*Clique Number*	*n*	*Clique Number*	*n*	*Clique Number*
2	1	35	12	68	20
3	2	36	12	69	20
4	3	37	12	70	20
5	3	38	13	71	20
6	4	39	13	72	21
7	4	40	13	73	21
8	5	41	13	74	22
9	5	42	14	75	22
10	5	43	14	76	22
11	5	44	15	77	22
12	6	45	15	78	22
13	6	46	15	79	22
14	7	47	15	80	23
15	7	48	16	81	23
16	7	49	16	82	23
17	7	50	16	83	23
18	8	51	16	84	24
19	8	52	16	85	24
20	9	53	16	86	24
21	9	54	17	87	24
22	9	55	17	88	24
23	9	56	17	89	24
24	10	57	17	90	25
25	10	58	17	91	25
26	10	59	17	92	25
27	10	60	18	93	25
28	10	61	18	94	25
29	10	62	19	95	25
30	11	63	19	96	25
31	11	64	19	97	25
32	12	65	19	98	26
33	12	66	19		
34	12	67	19		

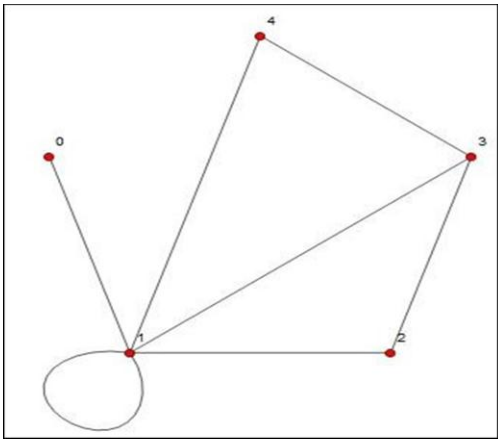

Fig. 1. Shown is the divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_5$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

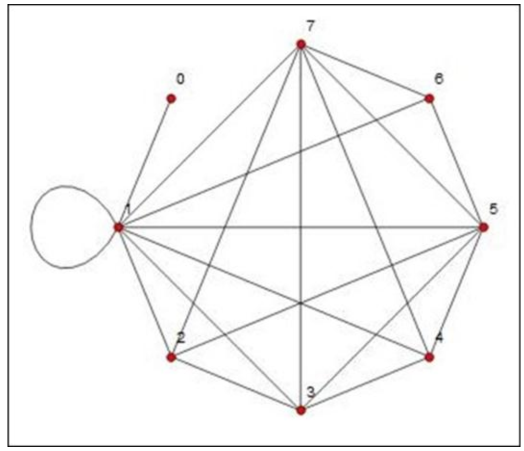

Fig. 2. Shown is the divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_8$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

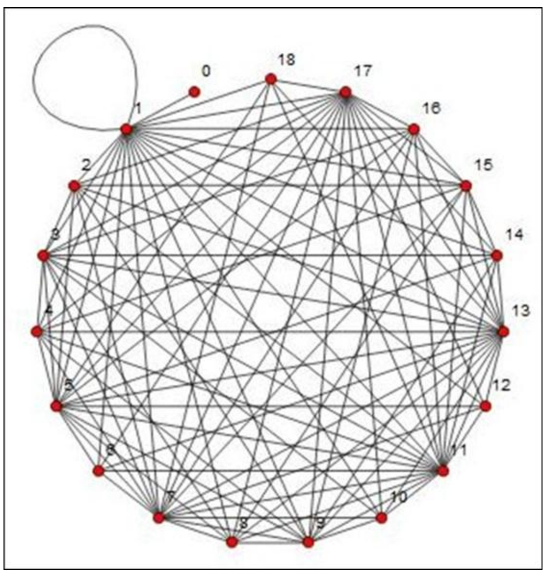

Fig. 4. Shown is the divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_{19}$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

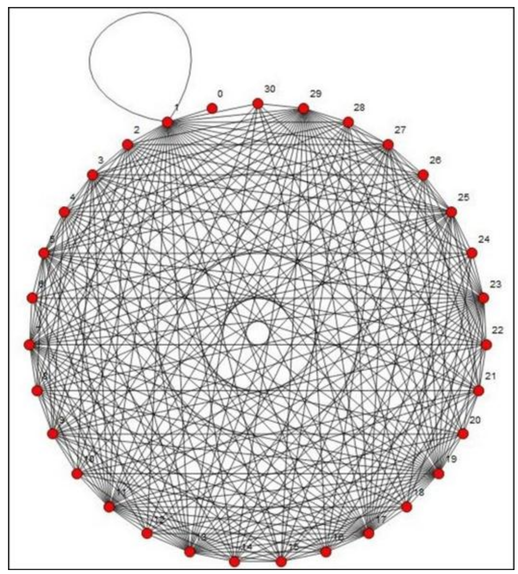

Fig. 3. Shown is the divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_{31}$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

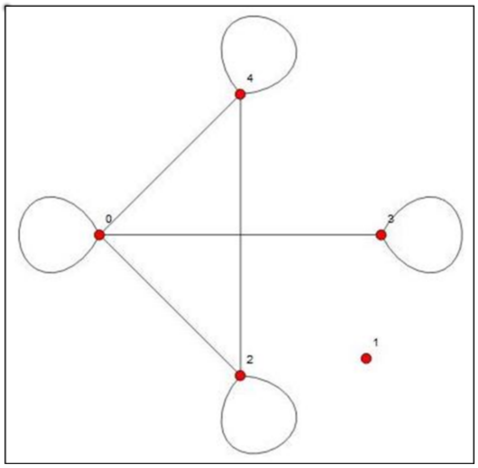

Fig. 5. Shown is the complement of divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_5$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

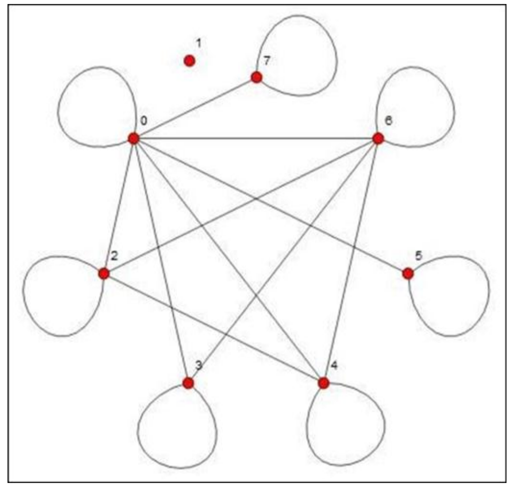

Fig. 6. Shown is the complement of divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_8$such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

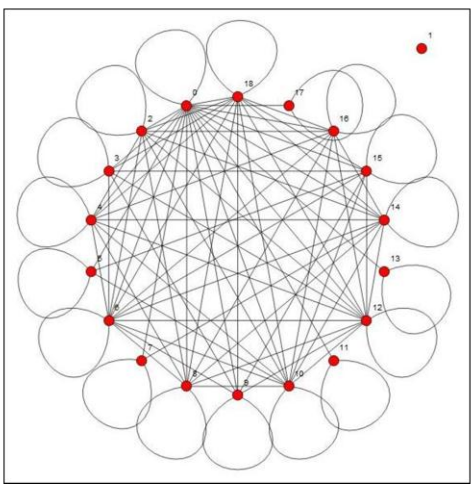

Fig. 7. Shown is the complement of divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_{19}$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

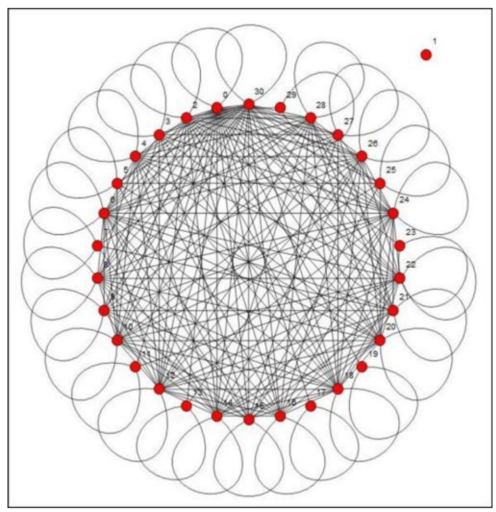

Fig. 8. Shown is the divisor graph $G(R)$ with vertices in the finite commutative ring $R = \mathbb{Z}_{31}$ such that, two distinct vertices u and v are adjacent if and only if $g.c.d.(u, v) = 1$.

IV- Conclusion:

In this work, we determined interesting properties of divisor graph. This work has provided some results on divisor graphs of the ring $R = \mathbb{Z}_n$. We think that it will be interesting for exploring other parameters of this graph and its complement, such as the maximum size of matching, circumference, chromatic number, acyclic chromatic number, acyclic chromatic index, etc.

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