On non-Continuous T-norms

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Abstract :

This paper provides a precise parametric classification of both left and right semicontinous T-norms along with the description of some features of T-norms without continuity assumptions.

Keywords : Partial order, continuity, binary relation

1- Introduction :

In fuzzy logics, T-norm means a binary operation $T: [0;1]^2 \rightarrow [0;1]$ satisfying the following axioms.

$(T1) T(y;x) = T(x;y) \leq T(\dot{x};y) \quad for\ any\ 0 \leq x \leq \dot{x}\ and\ 0 \leq y \leq 1$

$(T2) T\big(x;T(y;z)\big) = T(T(x;y);z) \quad for\ any\ 0 \leq x;y;z \leq 1$

$(T3)T(0;x) = 0 \; and \; T(1;x) = x \qquad for\; any\; 0 \leq x \leq 1$

Most authors[3,4,5] include also the continuity of T into the definition, and actually there is a complete classification for the continuous T-norms.

This paper focuses on the case of non-continous T-norms. In another terminology, (T1), (T2), (T3) mean that the algebraic structure $T :=$ $\left([0;1] \overset{T}{\bullet}, \geq\right)$ with the binary operation.

$$x \overset{T}{\bullet} y = T(x;y), x;y \in [0;1]$$

Is an ordered Abelian semigroup on [0;1] with neutral element 1 and sink 0. Also, in accordance with the usual terminology, we say that the T-norm $\overset{T}{\bullet}$ is strict if (T4) T(x_l;y)$< T(x_2;y)$ whenever $0 \leq x_1 < x_2 \leq 1 \; and \; 0 < y \leq 1$

In the sequel we shall fix an arbitrary T-norm^{T}, and we shall write simply xy instead of $x \overset{T}{\bullet} y$ without danger of confusion with the notation of the usual numerical product of real numbers (which may appear only as a simple special case of continuous T-norm). thus, in this terminology axioms (T1)… (T4) mean simply

$$(T1)xy = yx, \; (T2)x\,(yz) = (xy)z, (T3)0x = 0 \leq 1x = x, (T4)x_1y < x_2y\;(x_1 < x_2; y \neq 0)$$

We shall also use the customary notation a_n for the n-th $\overset{T}{\bullet}$-power $a^n = \underbrace{a \dots a}_{n\; terms}$ which is well – defined by the associativity (T_2).

Main Results:

1.1. Definition: The binary relations <≺ on the interval [0;1] is introduce as follows :

$$a \prec b \Leftrightarrow inf_n a^n \leq inf_n b^n;\ a \sim b :\Leftrightarrow a \prec b \prec a$$

1.2. Lemma. (1) the relation ≺ is a linear ordering with a ≺ b for a ≤ b. In particular,∼ is an equivalence relation whose equivalence classes are subintervals of [0,1].

(2) for any powr N we have a^N~a.

(3) We have ab ~ min{a , b}

Proof: As a consequence of axioms (T1) + (T3) the powers

$$T^{(n)}(x) = x^n \qquad (n = 1,2 \ldots)$$

are increasing function [0,1] → [0,1] with $T^{(1)} \geq T^{(2)} \geq T^{(3)} \geq \cdots$ Therefore their limit $T^{(\infty)}$ is a well-defined with

$$T^{(\infty)}(x) = inf_n x^n \quad (0 \leq x \leq 1)$$

By definition, we have a ≺ b iff $T^{(\infty)}(a) \leq T^{(\infty)}(b)$. Since the limit of increasing function is increasing.statement (1) is immediate.

(2) We have $T^{(\infty)}(a^N) = lim_n a^{Nn} = lim_n a^n = T^{(\infty)}(a)$.

(3) We may assume $a \leq b$ without loss of generality. Then $a^2 \leq ab \leq a1 = a$. Since $a \sim a^2$ by (2), and since the equivalence classes of ~ are intervals by (1), we conclude $a^2 \sim ab \sim a$ = min{a,b}.

Henceforth we introduce the notations

$$\mathfrak{T} := \{I_\propto : \propto \in A\} := \{\{x : x \sim a\} : a \in [0,1]\}$$

for the family of all equivalence classes of the relation ~. We know already that $\mathfrak{I}$ is a set of pairwise disjoint intervals forming a parrtions of [0,1] such that $I_\propto \leq I_\beta$ *and* $I_\propto \prec I_\beta$ (i.e. $a \prec b$ for all couples $(a, b) \in I_\propto \times I_\beta$) whenever $a \leq b$ for some $a \in I_\propto$ and $b \in I_\beta$. We shall say simply that the point $e \in [0,1]$ is an idempotent if it is idempotent with respect to the product $\overset{T}{\bullet}$, that is $e^2 = e \overset{T}{\bullet} e = T(e, e) = e$.

1.3. Corollary: (1) If the equivalence class $I_\propto$ is a left-closed interval then its initial point e :=min $I_\propto$ is an idempotent.

(2) If $\boldsymbol{I}_\propto$ is a non-degenrate right-closed interval then its endpoint f := max $\boldsymbol{I}_\propto$ is no idempotent, moreover $\boldsymbol{f} > \boldsymbol{f}^2 \geq \boldsymbol{f}^3 \geq \cdots . \to$ inf *I*.
(3) If $I_\propto$ is a non- degenerate right- open interval then f := sup $I_\propto$ is an idempotent.

(4) if $I_{\propto 1} < I_{\propto 2} < \cdots$ is an increasing sequence in $\mathfrak{I}$ then the point g :=sup$(\cup_n I_{\propto n})$ is an idempotent.

Proof: (1) Assume I = $\{x\colon x$~e$\}$ with e = min $I(\in I)$. Then $e = T^{(1)}(e) \geq T^{(2)}(e) = e^2$. By Lemma 1.2(2) we have $e^2 \sim e$ and hence $e^2 \in I$ with $e^2 \geq e$ =min *I*.

However, ingeneral $e = T^{(1)}(e) \geq T^{(2)}(e) = e^2$.

(2) Assume $I = \{x : x \sim e\}$ with f = max I $(\in I)$. Given any element $x \in I$, by definition we have $x \sim e$ with in$f_n x^n = T^{(\infty)}(x) = T^{(\infty)}(e)$. It follows

$$\inf\{x : x \sim f\} = \inf I = T^{(\infty)}(f)$$

Hence the case f = f^2 is impossible because this would imply inf $I = T^{(\infty)}(f)$ = f contradicting the non-degeneracy of I. Thus necessarily $f = 1f > f^2 = 1f^2 \geq f^3 \geq \to T^{(\infty)}(f) = \inf I$.

(3) Assume I = (x: x~e) with sup I = f($\notin I$). By lemma 1.2(3), the contrary f^2<f would imply the contraction f~f^2 with f ∈ I.

(4) Assume the contrary that let g > g^2. Then $g^2 < I_{an} <$ g for some index *n*. However, by Lemma 1.2(1)+(2), then we would have g^2 ~ x ~ g for all $x \in I_{an}$ entailing the contradiction $I_{an} > g \in I_{an}$.

1.4. Lemma. *Let P*: [0,1]→ [0,1] *be an increasing backward projection (that is p(y) ≤ P(x) = P(P(x)) ≤ x whenever 0 ≤ y ≤ x ≤ 1) onto the set* Ω. *Then the complement [0,1]/* Ω *is the union of a family of pairwise disjoint left-open intervals and*

$$P(x) = \max\,(\Omega \cap [0,x]) \qquad (x \in [0,1])$$

Proof: It suffices to see only that given any point $x \in [0,1]/\Omega$ with $P(x) < x$, every point y from the left-open interval $(P(x),x]$ is mapped into $P(x)$ by P. let $P(z) < y < z$ by assumption, P is an increasing mapping with $P = P \circ P$. Hence the conclusion $P(x) = P^2(x) \leq P(y) \leq P(x)$ entailing P(y) = P(x) is immediate. ■

1.5. Lemma. *Given a T-idempotent* $e = e^2 < 1$*, with its multiplication range* $\Omega_e := \{ex: x \in [0,1]\}$ *we have*

$$ex = \max(\Omega_e[0,x]) \qquad (0 \le x \le 1)$$

Also $e = \max \Omega_e$ *and* $[0,1] \setminus \Omega_e$ *is the union of a family of pairwise disjoint left-open intervals.*

Proof. According to (T1)+(T2), the mapping $P_e(x) := ex$ is an increasing backward projection of [0,1] onto Ω_e. indeed, $ey \le ex = (ee)x = e(ex)$ whenever $0 \le y \le x \le 1$. Since $w = Pe(w) \le Pe(1) = e \le \Omega_e$, necessary $e = \max \Omega_e$. The remaining statements are immediate from Lemma 1.4. ■

1.6 Lemma. *The set* E = {*idempotents*} *is left-closed that is* $E \ni e_n \nearrow e \Longrightarrow e \in E$.

Proof. Assume $E \ni e_n \nearrow e$.Then we have $e \ge e^2 \ge e_n^2 = e_n \nearrow e$ entailing $e = e^2 \in E$. ■

1.7. Proposition: *Let T be a strict T- norm. Then*

(1) the only idempotents are 0 and 1.

(2) We have $\{1\} = \{x: x \sim 1\}$ *and either* $I = \{|0,1|, \{1\}\}$*, or the interval* $\{x: x \sim 0\}$ *is closed with* $\max\{x: x \sim 0\} < 1$ *and each interval* $I_\alpha \in I$ *with* $0,1 \nexists I_\alpha$ *is non-degenrerate, open from left and closed from right.*

(3) there is no infinite strictly increasing sequence $I_{\alpha 2} < I_{\alpha 2} < \cdots$ *with sup* $(\cup_n I_{\alpha n}) < 1$ *in I.*

Proof:

(1) Assume $e \in (0,1)$ would be an idempotent. Then, by Lemma 1.5, we would have *ex = e for all* $e < x \leq 1$ contradicting the strictness of T.

(2) is immediate from statement (1) and Corollary 1.3(1)+(3).

(3) is immediate from from statement (1) and Corollary 1.3(4). ■

1.8.Corotlary. *Let T be a strict T-norm. Then there are two possibilities concerning the order structure of the family I of equivalence classes:*

(1) *I = {[0, 1), {1} };*

2- {x: x~0} = [0,w] with 0 < w < 1 and the interval (w, 1) can be decomposed to a sequence of intervals $(w_0, w_1]$, $(w_1, w_2]$,…, *with* $w_0 = w$ *and* $w_n \nearrow 1$ $(n \to \infty)$ *and each intervals* (w_n, w_{n+1}) *is covered by the disjoint union of a (necessarily countable) subfamily* $\{I_\alpha : \alpha \in A_n\}$ *of I being well-ordered by the relation <. Here order- consecutive intervals are joined at common endpoints.*

Recall that a function $\varphi: [0,1]^N \to [0,1]$ is said to be right [left] semiccontinu-ous if $\emptyset\,(x_n^{(1)} \ldots, x_n^{(1)}) \to \emptyset(x^{(1)}, \ldots, x^{(1)})$ whenever $x_n^{(1)} \searrow$ $x^{(1)} \ldots x_n^{(1)} \searrow x^{(1)}$ [res $x_n^{(1)} \nearrow x^{(1)}, \ldots, x_n^{(1)} \nearrow x^{(1)}$]. It is that if $\emptyset$ is increasing then the right [left] semi-continuity of all the sections $x \to \emptyset\,(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_N)$ implies the right [left] semicontinuity of $\emptyset$.

1.9. Lemma: *If N > 1 and $T^{(N)}$ is right semicontinous (in particular if T is right semicontinous) then all the intervals $I_\propto \in \mathfrak{T}$ are closed from left.*

Proof. Assume $T^{(N)}$ to be right semicontinuous and let $x \in I \in \mathfrak{T}$. Define e:= inf I and consider the sequence x^N, x^{2N}, x^{3N},…, By definition $x^n \searrow T^{(\infty)}(x) = e$. The right semicontinuity of $T^{(N)}$ entails $x^{nN} = T^N(x) \searrow T^{(N)}(e) = e^N$. However, since $(x^{nN})_{n=1}^{\infty}$ is a subsequence of $(x^n)_{n=1}^{\infty}$, we have $e = lim_n\, x^n = lim_n\, x^{Nn} = e^N$. since $e \geq e^2 \geq \ldots \geq e^N$ it follows $e^2 = e$ and hence $e \in I$ by Corollary 1.4. ■

1.10. Corollary. *If T is a right semicontinuous strict T-norm then* $\mathfrak{T}$= {[0,1), {1}}.

Proof. Immediate from Proposition 1.6 and Lemma 1.7. ■

****. Remark**. *Assuming the operation T to be continuous*, we can conclude the following .

(1) The powers $T^{(n)}$ (n = 1,2, …) are continuous increasing functions and hence their infimum $T^{(\infty)}$ is left semicontinuous and increasing.

(2) From (1) it readily follows that the intervals $I_\propto$ are closed from left with idempotent initial point.

(3) It is well-known that the idempotents of a continuous T-norm form a closed subset of [0,1] whose complement is the union of a countable family of pairwise disjoint open intervals. Hence one can deduce that the intervals Iα are either closed from the left and open from right or consist of a single

point which is necessarily an idempotent. The points of continuity of $T^{(\infty)}$ are exactly the idempotents of a continuous T-norm.

2- Structure Of A Equivalence Intervals :

Henceforth let S:= ([w,a],., ≥) be an ordered Abelian semigroup on the real intervals [w,a] such that :

(S1) $xy_1 \leq xy_2$ whenever $y_1 \leq y_2$.

(S2) $a > a^2 > a^3 > \ldots$ and $a^n \searrow w(n \to \infty)$

Since, by (S2), (w,a] is the disjoint union of the intervals $(a^{n+1}, a^n]$ $(n=1,2,\ldots)$, for any element $b \in (w,a]$ and for any index $k = 1,2 \ldots$ we can define

$$n_k(b) \coloneqq [n: a^{n+1} < b^k \leq a^n]$$

2.1. Lemma. *Given any* $b \in (w,a]$, *the intervals* $[n_k(b)/k, (n_k(b)+1)/k]$, $k=1,2,..$ have a unique common point.

Proof. Since for the lengths we have $|[n_k(b)/k, (n_k(b)+1)/k]| = 1/k \longrightarrow 0$ $(k \longrightarrow \infty)$, at most one common point may exists. To establish its existence, according to Helly's theorem. It suffices to see that each pair of them admits a non-empty intersection, that is

(2.2) $n_k(b)/k \leq (n_l(b)+1)/\ell$ for all $k, \ell = 1,2,\ldots.$

Consider any couple of incides $k \neq e$. by definition, $a^{n_k(b)+1} < b^k \leq a^{nk(b)}$ and hence by (S1), also $a^{\ell(nk(b)+1)} \leq b^{k\ell} \leq a^{\ell nk\ (b)}$. Similarly $a^{k(n\ell(b)+1)} \leq b^{k\ell}$ $\leq a^{kn\ell(b)}$. It follows $a^{k(ne(b)+1)} \leq b^{k\ell} \leq a^{\ell nk(b)}$ and hence by (S2) we conclude $k(n_\ell(b)+1 \geq \ell n_k(b)$ which is equivalent to (2.2). ■

2.3. Definition. Henceforth we write

$L(b) \coloneqq [\textit{the unique of } \bigcap_{k=1}^{\infty}[n_k(b)/k, (n_k(b)+1)/k]]$ for any $b \in (w,a]$

Furthermore Λ:= L((w,a]) shall denote the range of the function L.

2.4. Remarks : (1)$n_k(b) \in [\lceil kL(b) \rceil - 1, \lceil kL(b) \rceil + 1]$ for all $k =$ $1,2,..$ and $b \in (w,a]$. *

(2) If $b \in (a^{n+1}, a^n)$then $L(b) \in [n, n+1]$. In particular $L(a^n)$

$= n, (n = 1,2,..)$

(3) the mapping L is decreasing trivially, but not necessarily strictly decreasing

Example: $S \coloneqq ((-\infty, 1], .., \geq)$ with $xy \coloneqq \lceil x \rceil + \lceil y \rceil$and $L(b) = \lfloor b \rfloor$

2.5. Lemma. *We have* $L(bc) = L(b) + L(c)$ for all $b,c \in (w,a]$.

Proof. According to Remark 2.4(1), $L(bc) = \lim_{k\to\infty} n_k(bc)/k$. By definition, $a^{nk(b)} \geq b^k > a^{n_k(b)+1}$ and $a^{n_k(c)} \geq c^k > a^{n_k(c)+1}$. Hence $a^{n_k(b)+n_k(c)} \geq (bc)^k \geq a^{n_k(b)+n_k(c)+2}$. By the definition of the value n_k (bc) and axiom (S2) it follows $n_k(b) + n_k(c) - 1 \leq n_k(bc) \leq n_k(c) + 3$.

* $\lceil x \rceil \coloneqq \inf\{n \text{ integer} : x \leq n\}$ standing for the upper entier part function.

Therefore $L(b)+L(c)=\lim_{k\to\infty}\left(n_k(b)+n_k(c)\right)/k=\lim_{k\to\infty} n_k(bc)/k=L(bc)$. ■

2.6. Corollary :

(1) *The range* Λ of L *is a subsemigroup of* $([1,\infty),+)$.

(2) *In particular* Λ *is countable under the hypothesis that L is not strictly decreasing and (S1*)* $xy_1 < xy_2$ whenever $y_1 \leq y_2$.

(3) Λ *is Lebesgue – measurable. If it has positive Lebsgue measure, for some n we have* $[n,\infty) \subset \Lambda$.

Proof.

(1) Is immediate from Lemma 2.5.

(2) The inverse Image $L^{-1}\{\xi\} := \{b: L(b) = \xi\}$, $\xi \in \Lambda$ are pairwise disjoint intervals since the function L is decreasing. If L is not strictly decreasing, some interval $L^{-1}\{\xi_0\}$ has positive length. By 1.5 we have $L^{-1}\{\xi_0+\eta\} \supset L^{-1}\{\xi_0\} + L^{-1}\{\eta\}$ and $L^{-1}\{\xi_0+n\}$ is also a non-degenerate interval for any $\eta \in \Lambda$ if (S1*) holds. Since there may only be countably many pairwise disjoint non-degenerate real intervals, we conclude (2).

(3) It is well-known that the rnage of a decreasing real function is a Borel set (actually a sequence of points added to an interval minus a countable union of intervals). In particular Λ = range (L) is Borel measurable. Suppose mes $(\Lambda) > 0$ (mes denoting Lebsegue measure) then almost every point of Λ is Lebsegue point. In pariturclar , mes $(\Lambda \cap [\propto,\beta] > (\beta - a)/2$

for some $1 \leq \propto < \beta$. Recall that given any set Ω of real numbers with density $> 1/2$, the sum $\Omega + \Omega := \{\omega_1 + \omega_2 : \omega_1, \omega_2 \epsilon\, \Omega\}$ contains an interval with positive length.* Hence we conclude that $\Lambda \supset \Lambda + \Lambda \supset (\Lambda \cap [\propto, \beta]) + (\Lambda \cap [\propto, \beta])$ contains some interval I of length $\delta > 0$. It is immediate that $\Lambda \supset \Lambda + \cdots + \Lambda$ with $\lceil 1/\delta \rceil$ terms contains the interval $J := I + \cdots + I$ with length >1. According to Remark 2.4(2), we have $\{1,2,\ldots\} \subset \Lambda$. It follows $\Lambda \supset \bigcup_{k=0}^{\infty} k + J \supset [\lceil \inf J \rceil, \infty)$. ■

2.7 *Lemma. (1) If the underlying product is left semicontinuous* $[i.e. \dot{x}_i y \nearrow xy \text{ whenever } x_i \nearrow x]$ *then its logarithm L is also left semicontinuous.*
(2) *If the product is right semicontinuous then L is right semicontinuous.*

Proof: Assume the product is left semicontinuous. It is well-Known that then we have even $x_i y_i \nearrow xy$ whenever $x_i \nearrow x$ and $y_i \nearrow y$.

(Indeed, given any $\varepsilon > 0$, there exists jo with $xy \geq xy_{jo} \geq xy - \varepsilon/2$. Also there exists $j_1 \geq j_0$ with $xy_{jo} \geq x_{j1} y_{jo} \geq xy_{jo} - \varepsilon/2$ and hence $xy \geq x_{j1} y_{jo} \geq xy - \varepsilon$. Given any couple $x_i \nearrow x$ resp. $y_i \nearrow y$ of sequences, for any $i \geq j_1$ we have $xy \geq x_i y_i \geq x_{j1} x_{j0} \geq xy - \varepsilon$). In particular the powers $b \mapsto b^k (k = 1,2, \ldots)$ are left semicontinuous. It follows that, for any fixed k, the step function $b \longmapsto n_k(b)$ is left semicontinuous. Prrof : Fix k arbitrarily. Since the power $b \longmapsto b^k$ is increasing, the function, n_k

*Proof. We may assume
$\Omega \supset [\propto, \beta] \setminus \bigcup_{k=1}^{\infty} I_k$ where $I_1, I_2, \ldots$ are pairwise disjoint open intervals with $\sum_{k=1}^{\infty} mes\,(I_k) = (\beta - \propto)(1/2 - \varepsilon)$ for some $\varepsilon > 0$. The vertical resp.
Horizontal stripes $I_k \times [\propto, \beta]$and $[\propto, \beta] \times I_k, k = 1,2, \ldots$ cut most $2(1/2\text{-}\varepsilon)\sqrt{2}(\beta - \propto)$ length from the diagonal segments $D_p := \{(\omega_1 . \omega_2) : \propto \leq \omega_1, \omega_2 \leq \beta, \omega_1 + \omega_2 = p\}$ which have length $> \sqrt{2}(\beta - \propto - \varepsilon)$ whenever $p \in (\propto + \beta - \varepsilon, \propto + \beta + \varepsilon). Therefore\ \Omega + \Omega \supset (\propto + \beta - \varepsilon, \propto + \beta + \varepsilon)$.

(.)decreases.Consider a sequence $b_i \nearrow b > \omega$. Since $\omega < inf_i\, b_i \leq a$, the decreasing sequence $\{n_k(b_i): i = 1,2,\dots\}$ is bounded. Since n_k (.) Assumes integer values, there is

i_0 with $n_k(b_i) = N := lim_i\, n_k(b_i)\ for\ i \geq i_0$. Then $a^{N+1} = a^{n_k(b_i)-1} < b_i^k \leq a^{n_k(b_i)} = a^N$ for any $i \geq i_0$. It follows $a^{N+1} > b \geq a^N$ which means that $n_k(b) = N$ $i.e.\ n_k(b_i) \nearrow N = n_k(b)$. On the other hand the sequence $n_k(.)/k(k = 1,2,\dots)$ converges uniformly to L (.) (actually $sup_b\, |L(b) - n_k(b)/k| \leq 1/k$ for all k). Hence we deduce that left semicontinuity of L, because, in general, the uniform limit of τ-contyinuous functions is τ-continuous for any topology τ.Thus,in particular L is left semicontinuous.The proof

of (2) is analogous with the step functions $\tilde{n}_k(b) \coloneqq [n: a^n \leq b^k < a^{n-1}]$ in place of n_k (.). ■

2.8 Lemma. For any $c \in (\omega, a]$, $the\ functions\ n_k^c(b) \coloneqq [n : c^{n+1} < b^k \leq c^n] and\ L^c(b) \coloneqq lim_k n_k^c(b)/k\ \ are\ well-defined, moreover\ we\ have\ L^c = L\ (c)^{-1}\ L\ in\ terms\ of\ the\ logarithm\ function\ defined\ in$ 1.3.

Proof. S^c = ((w,c],…≥) is an orderd subsemigroup of S = ((w,a],..≥). Hence we can apply the previous arguments with c in place of a to establish that all the function n^c along with L^c are well-defined and decreasing. By definition we have $c^{n}{}_{k}{}^{n(b)+1} < b^k < c^{n}{}_{k}{}^{c(b)}$, whence

$(n_k^c(b) + 1)\ L(c) = L(c^{nk(b)+1}) > L(b^k) = kL(b) > L(c^{nk(b)}) = n_k^c(b)\ L(c)$.

Since $L^c(b) = \lim_k n_k^c(b)/k$, we get $L^c(b)\ L(c) > L(b) > L^c(b)\ L(c)$. ■

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